# CONTROL OF TANDEM-TYPE TWO-WHEEL VEHICLE AT VARIOUS NOTION MODES ALONG SPATIAL CURVED LAY OF LINE

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*Wheeled vehicle is considered as a material point under the conditions of non-uniform movement along curved spatial lay of line. Hodograph in a class of spiral lines describes kinematics of a vehicle. A kinetostatics problem of tandem-type two-wheel vehicle is being solved. Equivalent contact dynamics are being determined.*

*Keywords: hodograph, kinematics, kinetostatics, Euler-Lagrange equations, equivalent force systems, invariants of statics.*

**Introduction.** A problem of dynamic design in the context of two-wheel vehicle controllability as well as dynamic burden of its design and road surface is important in terms of various motion modes (accelerated, decelerated, and steady) along curved spatial lay of line in junctions, at various gradients, on straight and turns as well as within other curved areas [1, 5, 4, 7]. The problem solution will help determine equivalent contact control force making analysis of the required control facilities.

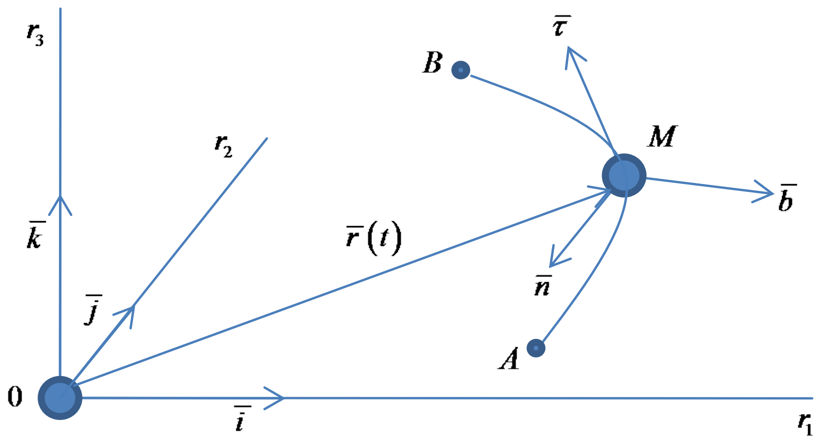
**The problem statement.** Hodograph of vehicle motion along curved spatial lay of line is supposed as given. A vehicle is considered as material point of known mass moving under the gravity, given aerodynamic forces, and searched equivalent contact moving (control) forces – resulting reaction of contact with reference trajectory (lay of line). Curved spatial reference trajectory that is lay of line is identified by hodograph in motionless earth reference.

**Hodograph** correspondent to real trajectory of vehicle motion [7] in a class of spiral lines [5] is specified in motionless (earth) reference as follows

$$\bar{r}(t) = \left\| \rho_0 \rho_1 \rho_2 \rho_3 \right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| \left( \bar{i} \cos \omega t + \bar{j} \sin \omega t \right) + \bar{k} \left\| h_0 h_1 h_2 h_3 \right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\|$$

where $\rho_i\, h_i\,(i = 0,1,2,3)$ are varied parameters determined on specified boundary conditions; $\omega$ is mean angular turn velocity equal to $\omega = \frac{\varphi_0}{t_0}$. Here $\varphi_0$ is complete turn angle; and $t_0$ is required time of turn passing.

Following figure demonstrates lay of line as a trajectory of wheeled vehicle motion in terms of curved area being adequate to proposed hodograph:




Here $\overline{i}, \overline{j}, \overline{k}$ are orts of earth (motionless) reference; and

$\overline{\tau}, \overline{n}, \overline{b}$ are orts of movable, natural axes.

It is obvious that hodograph is represented in a well-known representation form :

$$\overline{r}(t) = \overline{i}\, r_1 + \overline{j}\, r_2 + \overline{k}\, r_3 .$$

Here vector components are assumed as:

$$r_1 = \left\|\rho_0 \rho_1 \rho_2 \rho_3\right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| \cos\omega t\,,\ \ r_2 = \left\|\rho_0 \rho_1 \rho_2 \rho_3\right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| \sin\omega t\,,\ \ r_3 = \left\|h_0 h_1 h_2 h_3\right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| .$$

Here you can find hodograph of a vehicle motion:

1. steady $(V_{1A} = V_{1B})$ motion within horizontal $(h_j = 0;\ j = 0, 1, 2, 3)$ straight $(\omega = 0)$ lay of line:

$$\overline{r}(t) = \overline{i}\,(r_{1A} + V_{1A} \cdot t)\,;$$

2. unsteady: $(V_{1A} < V_{1B})$ –accelerated; $(V_{1A} > V_{1B})$ – decelerated motion within horizontal $(h_j = 0;\ j = 0, 1, 2, 3)$ straight $(\omega = 0)$ lay of line.

3. steady $(V_{1A} = V_{1B},\ V_{3A} = V_{3B} = 0\,)$ motion within profile-inclined lay of line if $(r_{3A} < r_{3B})$ – rise and $(r_{3A} > r_{3B})$ – incline:

$$\overline{r}(t) = \overline{i}\,(r_{1A} + V_{1A} \cdot t) + \overline{k}\left(3 - 2\frac{V_{1A}}{r_{1B} - r_{1A}} \cdot t\right)\left(\frac{V_{1A}}{r_{1B} - r_{1A}}\right)^2 r_{3B} \cdot t^2 .$$

Here using Cartesian coordinate system lay of line profile is represented in the form of square and cubic parabolas: $z = 3x^2 - 2x^3$, where

$$x = \frac{r_1(t) - r_{1A}}{r_{1B} - r_{1A}}, \;\; z = \frac{r_3(t)}{r_{3B}}.$$

4. Unsteady motion within horizontal plane where direct- angle turn takes place:

$$\overline{r}(t) = \left[ r_{1A} + \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 (r_{2B} - r_{1A})t^2 - \frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 (r_{2B} - r_{1A})t^3 \right] \cdot \left( \overline{i}\cos\frac{V_{2A}}{r_{1A}}t + \overline{j}\sin\frac{V_{2A}}{r_{1A}}t \right).$$

Here lay of line plan in polar coordinate system is represented by square and cubic Archimedean spirals:

$$\frac{r(\varphi) - r_{1A}}{r_{2B} - r_{1A}} = 3\left(\frac{\varphi}{\pi/2}\right)^2 - 2\left(\frac{\varphi}{\pi/2}\right)^3,$$

where polar angle is: $\varphi = \omega t$ ; and

polar radius is: $r(\varphi) = r_1^2 + r_2^2$ .

Moreover, when $V_{2A} = V_{1B}$ or $r_{1A} = r_{2B}$ , it follows that: $r(\varphi) = r_{1A}$ at any $\varphi$ , i.e. we obtain lay of line in the form of radial arc within the given interval: $0 \le \varphi \le \frac{\pi}{2}$ .

**Kinematics.** Vector of vehicle linear velocity in the form of material point is determined on the specified hodograph as:

$$\overline{V} = \frac{d\overline{r}}{dt} \text{ or } \overline{V} = \overline{i}\,\dot{r}_1 + \overline{j}\,\overline{r_2} + \overline{k}\,\dot{r}_3 .$$

Velocity value is determined with the help of scalar product:

$$\overline{V} \cdot \overline{V} = \upsilon^2 \text{ or } \upsilon^2 = \dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2 .$$

By definition, velocity value is also determined as a time derivative from the path:

$$\upsilon = \frac{ds}{dt} \text{ or } \upsilon = \dot{S} .$$

Then the path of a vehicle within random time period is calculated by means of definite integral with variable upper limit:

$$S(t) = \int_0^t \upsilon(t)\,dt \text{ or } S(t) = \int_0^t \sqrt{\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2}\,dt .$$

When path is introduced as intermediate argument, velocity vector is represented as:

$$\bar{V} = \frac{d\bar{r}}{ds}\frac{ds}{dt} \text{ or } \bar{V} = \frac{d\bar{r}}{ds}\dot{S}$$

And taking into account that: $\frac{d\bar{r}}{ds} = \bar{\tau}$ , we obtain [6]:

$$\bar{V} = \bar{\tau}\,\dot{S}\ .$$

It is obvious that velocity vector projection on tangent line ort to spatial trajectory is $\bar{\tau}\cdot\bar{V} = \bar{\tau}\cdot\bar{\tau}\,\dot{S}$ . Determine velocity value as follows: $V_\tau = \dot{S}$ ; to principal normal ort: $\bar{n}\cdot\bar{V} = \bar{n}\cdot\bar{\tau}\,\dot{S}$ , $V_n = 0$ ; to binormal ort: $\bar{b}\cdot\bar{V} = \bar{b}\cdot\bar{\tau}\,\dot{S}$ , $V_b = 0$ . In terms of vector and matrix form we obtain:

$$\begin{Vmatrix}\bar{\tau}\ \bar{n}\ \bar{b}\end{Vmatrix}\begin{Vmatrix}V_\tau\\ V_n\\ V_b\end{Vmatrix} = \begin{Vmatrix}\bar{i}\ \bar{j}\ \bar{k}\end{Vmatrix}\begin{Vmatrix}\dot{r}_1\\ \dot{r}_2\\ \dot{r}_3\end{Vmatrix} \text{, or } \bar{\tau}\,\dot{S} = \bar{i}\,\dot{r}_1 + \bar{j}\,\dot{r}_2 + \bar{k}\,\dot{r}_3\,.$$

It is known that scalar and vector productions of vectors are represented in quaternion matrices. Then in earth reference we define:

$$\begin{Vmatrix}\dfrac{\bar{V}\cdot\bar{V}}{0}\end{Vmatrix} \leftrightarrow \frac{1}{2}\left(\dot{R}_0 + \dot{R}_0^t\right)\dot{r}_0\,,$$

where $\dot{R}_0 = \begin{Vmatrix}0 & \dot{r}_1 & \dot{r}_2 & \dot{r}_3\\ -\dot{r}_1 & 0 & -\dot{r}_3 & \dot{r}_2\\ -\dot{r}_2 & \dot{r}_3 & 0 & -\dot{r}\\ -\dot{r}_3 & -\dot{r}_2 & \dot{r}_1 & 0\end{Vmatrix}$ , $\dot{R}_0^t = \begin{Vmatrix}0 & \dot{r}_1 & \dot{r}_2 & \dot{r}_3\\ -\dot{r}_1 & 0 & \dot{r}_3 & -\dot{r}_2\\ -\dot{r}_2 & -\dot{r}_3 & 0 & \dot{r}_1\\ -\dot{r}_3 & \dot{r}_2 & -\dot{r}_1 & 0\end{Vmatrix}$ , $\dot{r}_0 = \begin{Vmatrix}0\\ \dot{r}_1\\ \dot{r}_2\\ \dot{r}_3\end{Vmatrix}$ ,

or

$$\begin{Vmatrix}\dfrac{\bar{V}\cdot\bar{V}}{0}\end{Vmatrix} \leftrightarrow \begin{Vmatrix}0 & \dot{r}_1 & \dot{r}_2 & \dot{r}_3\\ -\dot{r}_1 & 0 & 0 & 0\\ -\dot{r}_2 & 0 & 0 & 0\\ -\dot{r}_3 & 0 & 0 & 0\end{Vmatrix}\begin{Vmatrix}0\\ \dot{r}_1\\ \dot{r}_2\\ \dot{r}_3\end{Vmatrix},$$

i.e.

$$\begin{Vmatrix}\dfrac{\bar{V}\cdot\bar{V}}{0}\end{Vmatrix} \leftrightarrow \begin{Vmatrix}\dfrac{\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2}{0}\end{Vmatrix}.$$

Similarly in natural axes we obtain:

$$V_0 = \begin{Vmatrix}0 & V_\tau & V_n & V_b\\ -V_\tau & 0 & -V_b & V_n\\ -V_n & V_b & 0 & V_\tau\\ -V_b & -V_n & V_\tau & 0\end{Vmatrix},\ V_0^t = \begin{Vmatrix}0 & V_\tau & V_n & V_b\\ -V_\tau & 0 & V_b & -V_n\\ -V_n & -V_b & 0 & V_\tau\\ -V_b & V_n & -V_\tau & 0\end{Vmatrix},\ \upsilon_0 = \begin{Vmatrix}0\\ V_\tau\\ V_n\\ V_b\end{Vmatrix},$$

$$\begin{Vmatrix}\dfrac{\bar{V}\cdot\bar{V}}{0}\end{Vmatrix} \leftrightarrow \begin{Vmatrix}0 & V_\tau & V_n & V_b\\ -V_\tau & 0 & 0 & 0\\ -V_n & 0 & 0 & 0\\ -V_b & 0 & 0 & 0\end{Vmatrix}\begin{Vmatrix}0\\ V_\tau\\ V_n\\ V_b\end{Vmatrix} \text{ or } \begin{Vmatrix}\dfrac{\bar{V}\cdot\bar{V}}{0}\end{Vmatrix} \leftrightarrow \begin{Vmatrix}0 & \dot{S} & 0 & 0\\ -\dot{S} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{Vmatrix}\begin{Vmatrix}0\\ \dot{S}\\ 0\\ 0\end{Vmatrix} \text{ i.e. } \begin{Vmatrix}\dfrac{\bar{V}\cdot\bar{V}}{0}\end{Vmatrix} \leftrightarrow \begin{Vmatrix}\dfrac{\dot{S}^2}{0}\end{Vmatrix}.$$

Obvious equality follows:

$$\dot{S}^2 = \dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2 \,.$$

In the context of earth reference, linear acceleration vector is found out according to the specified hodograph in the form of:

$$\overline{W} = \frac{d^2\overline{r}}{dt^2} \text{ or } \overline{W} = \overline{i}\,\ddot{r}_1 + \overline{j}\,\ddot{r}_2 + \overline{k}\,\ddot{r}_3 \,.$$

Acceleration velocity is determined with the help of scalar production:

$$\overline{W} \cdot \overline{W} = \ddot{r}_1^2 + \ddot{r}_2^2 + \ddot{r}_3^2 \,.$$

Linear acceleration vector in natural axes is [6]:

$$\overline{W} = \overline{\tau}\ddot{S} + \overline{n}\,K\,\dot{S}^2 \,,$$

i.e.

$$W_\tau = \overline{\tau} \cdot \overline{W} \text{ or } W_\tau = \ddot{S} \,,$$
$$W_n = \overline{n} \cdot \overline{W} \text{ or } W_n = K\,\dot{S}^2 \,,$$
$$W_b = \overline{b} \cdot \overline{W} \text{ or } W_b = 0 \,,$$

where $K$ is curvature.
Then

$$\overline{W} \cdot \overline{W} = \ddot{S}^2 + K^2\,\dot{S}^4 \,,$$

i.e. dependence with specified hodograph takes place:

$$\ddot{S}^2 + K^2\,\dot{S}^4 = \ddot{r}_1^2 + \ddot{r}_2^2 + \ddot{r}_3^2 \,.$$

Hodograph also determines $W_\tau$ and $W_n$ components of following closed vector form:

$$W_\tau = \frac{\dot{\overline{r}} \cdot \ddot{\overline{r}}}{\left|\dot{\overline{r}}\right|} \,, \qquad W_n = \frac{\left|\dot{\overline{r}} \times \ddot{\overline{r}}\right|}{\left|\dot{\overline{r}}\right|} \,.$$

Here scalar and vector products are convenient to be calculated in quaternion matrices [3]:

$$\left\| \frac{\dot{\overline{r}} \cdot \ddot{\overline{r}}}{0} \right\| \leftrightarrow \frac{1}{2}\left(\dot{R}_0 + \dot{R}_0^t\right)\ddot{r}_0 \,, \quad .$$

**Curvature** in the system of natural trihedral coordinates is determined as:

$$K^2 = \frac{W_n^2}{\dot{S}^4} \,,$$

Where

$$W_n^2 = \bar{W} \cdot \bar{W} - W_\tau^2 .$$

Tangential acceleration in coordinate system is determined according to the formula:

$$W_\tau = \frac{1}{\dot{S}}\left(\bar{V} \cdot \bar{W}\right) \text{ or } W_\tau = \frac{\dot{r}_1 \ddot{r}_1 + \dot{r}_2 \ddot{r}_2 + \dot{r}_3 \ddot{r}_3}{\left(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2\right)^{\frac{1}{2}}} .$$

Then

$$W_n^2 = \ddot{r}_1^2 + \ddot{r}_2^2 + \ddot{r}_3^2 - \frac{\left(\dot{r}_1 \ddot{r}_1 + \dot{r}_2 \ddot{r}_2 + \dot{r}_3 \ddot{r}_3\right)^2}{\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2} .$$

Consequently in earth reference we obtain:

$$K^2 = \frac{\left(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2\right)\left(\ddot{r}_1^2 + \ddot{r}_2^2 + \ddot{r}_3^2\right) - \left(\dot{r}_1 \ddot{r}_1 + \dot{r}_2 \ddot{r}_2 + \dot{r}_3 \ddot{r}_3\right)^2}{\left(\dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2\right)^3} .$$

Equivalent vector form provides closed curvature record [4] :

$$K^2 = \frac{\left(\dot{\bar{r}} \times \ddot{\bar{r}}\right) \cdot \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right)}{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)^3} \text{ or } K^2 = \frac{\dot{\bar{r}} \cdot \left[\ddot{\bar{r}} \times \left(\dot{\bar{r}} \times \ddot{\bar{r}}\right)\right]}{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)^3} .$$

Using determinant we also find out:

$$K^2 = \frac{\begin{vmatrix} \dot{\bar{r}} \cdot \dot{\bar{r}} & \dot{\bar{r}} \cdot \ddot{\bar{r}} \\ \ddot{\bar{r}} \cdot \dot{\bar{r}} & \ddot{\bar{r}} \cdot \ddot{\bar{r}} \end{vmatrix}}{\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)^3} .$$

Quaternion matrices provide convenient curvature calculation where $\ddot{r}_0^t = \left\| 0 \ \ddot{r}_1 \ \ddot{r}_2 \ \ddot{r}_3 \right\|$.

Here first and second products of its time components for hodograph under consideration $\bar{r}(t)$ are determined as follows:

$$\dot{r}_1 = \left\|\rho_0 \rho_1 \rho_2 \rho_3\right\| \left[ \begin{Vmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{Vmatrix} \cos\omega t - \omega \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \sin\omega t \right], \quad \dot{r}_2 = \left\|\rho_0 \rho_1 \rho_2 \rho_3\right\| \left[ \begin{Vmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{Vmatrix} \sin\omega t + \omega \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \cos\omega t \right], \quad \dot{r}_3 = \left\|h_0 h_1 h_2 h_3\right\| \begin{Vmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{Vmatrix}$$

$$\ddot{r}_1 = \left\|\rho_0 \rho_1 \rho_2 \rho_3\right\| \left[ \begin{Vmatrix} -\omega^2 \\ -\omega^2 t \\ 2-\omega^2 t^2 \\ 6t-\omega^2 t^2 \end{Vmatrix} \cos\omega t - 2\omega \begin{Vmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{Vmatrix} \sin\omega t \right], \quad \ddot{r}_2 = \left\|\rho_0 \rho_1 \rho_2 \rho_3\right\| \left[ \begin{Vmatrix} -\omega^2 \\ -\omega^2 t \\ 2-\omega^2 t^2 \\ 6t-\omega^2 t^2 \end{Vmatrix} \sin\omega t + 2\omega \begin{Vmatrix} 0 \\ 1 \\ 2t \\ 3t^2 \end{Vmatrix} \cos\omega t \right],$$

$$\ddot{r}_3 = \left\| h_0 h_1 h_2 h_3 \right\| \left\| \begin{matrix} 0 \\ 0 \\ 2 \\ 6t \end{matrix} \right\| .$$

**Kinetics.** Mathematical model of two-wheel vehicle in terms of its spatial motion along curved lay of line is developed using non-linear differential Euler-Lagrange equations in the form of quaternion matrices [3]. Projections of vehicle velocity vector on natural axes are assumed as quasi-velocities. Natural trihedral is taken as a bound coordinate system which pole is combined with the material point assumed as a vehicle model. Two-wheel vehicle is considered as the material point of the known mass with the applied inertial forces, gravitation force, aerodynamic forces, and the required contact moving (control) forces ensuring necessary motion mode along the specified spatial curved lay of line. Then kinetostatics equations are as follows [2]:

$$\frac{1}{m} \left\| \begin{matrix} 0 \\ N_\tau \\ N_n \\ N_b \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ W_\tau \\ W_n \\ 0 \end{matrix} \right\| + g\, A^t \cdot {}^t A^t \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\| - \frac{qS}{m} R_d \cdot {}^t R_d \left\| \begin{matrix} 0 \\ c_{1d} \\ c_{2d} \\ c_{3d} \end{matrix} \right\| .$$

where $m$ is vehicle mass; $g$ is gravity acceleration;

$q$ is velocity pressure; $S$ is characteristic area;

$c_{1d}, c_{2d}, c_{3d}$ are aerodynamic coefficients;

$W_\tau, W_n$ are quasi-accelerations;

$A$ is quaternion matrix in terms of Rodriguez-Hamilton parameters determining orientation of natural trihedral in earth reference;

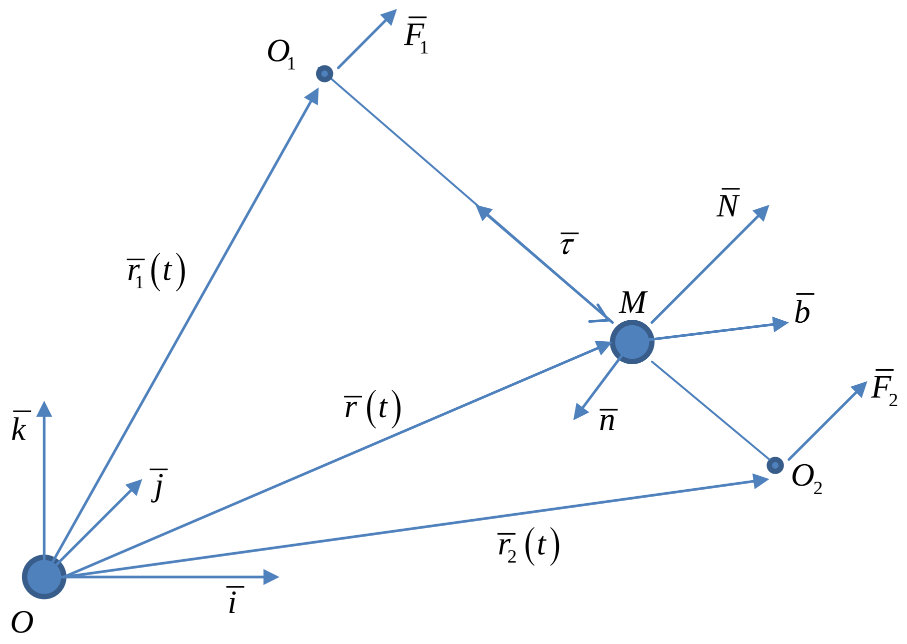
$R_d$ is quaternion matrix determining orientation of aerodynamic axes relative to natural ones; and

$N_\tau, N_n, N_b$ are moving forces.

Kinematic correlations in quaternion matrices closing the given kinetostatics equations are as follows [3]:

$$\left\| \begin{matrix} 0 \\ \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{matrix} \right\| = A \cdot {}^t A \left\| \begin{matrix} 0 \\ V_\tau \\ 0 \\ 0 \end{matrix} \right\|, \qquad \left\| \begin{matrix} 0 \\ V_\tau \\ 0 \\ 0 \end{matrix} \right\| = A^t \cdot {}^t A^t \left\| \begin{matrix} 0 \\ \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{matrix} \right\| .$$

**Statics.** Obtained resulting moving force $(\bar{N})$ which provides motion of two-wheel vehicle along specified lay of line in the known mode is represented as a system of two equivalent contact control forces $(\bar{F}_1, \bar{F}_2)$ to be determined. Following figure shows them:




Here reference points are given in movable natural axes: $O_1M = l_1$, $O_2M = l_2$.
Then according to Varignon theorem [6] we obtain:

$$\bar r_1 \times \bar F_1 + \bar r_2 \times \bar F_2 = \bar r \times \bar N\,,$$

where $\bar r_1 = \bar r + \bar\tau l_1$, $\bar r_2 = \bar r - \bar\tau l_2$.

In particular, if $\bar r_2 = 0$, then $\bar r = \bar\tau l_2$, $\bar r_1 = (l_1 + l_2)\bar\tau$ and $(l_1 + l_2)\bar\tau \times \bar F_1 = l_2 \bar\tau \times \bar N$.

Hence:

$$F_{2n} = \frac{l_1}{l_1 + l_2} N_n\,, \qquad F_{2b} = \frac{l_1}{l_1 + l_2} N_b\,,$$

and

$$\frac{F_{2n}}{N_n} = \frac{F_{2b}}{N_b} = \frac{l_1}{l_1 + l_2}$$

Or parallelism condition:

$$\frac{F_{2n}}{F_{2b}} = \frac{N_n}{N_b}\,.$$

Consequently:

$$\frac{F_{1n}}{F_{1b}} = \frac{F_{2n}}{F_{2b}} = \frac{N_n}{N_b}\,.$$

If $\bar r = 0$, then $\bar r_1 = l_1 \bar\tau$, $\bar r_2 = -l_2 \bar\tau$ and $l_1 \bar\tau \times \bar F_1 - l_2 \bar\tau \times \bar F_2 = 0$.
It follows:

$$\frac{F_{1b}}{F_{2b}} = \frac{l_2}{l_1}\,, \qquad \frac{F_{1n}}{F_{2n}} = \frac{l_2}{l_1}$$

or

$$\frac{F_{1n}}{F_{2n}} = \frac{F_{1b}}{F_{2b}} = \frac{l_2}{l_1},$$

i.e. pre-determined parallelism condition

$$\frac{F_{1n}}{F_{1b}} = \frac{F_{2n}}{F_{2b}}.$$

**Static invariants** are required to verify obtained results. In particular, within normal plane of natural trihedral static invariant one:

$$F_{1n} + F_{2n} = N_n, \qquad F_{1b} + F_{2b} = N_b$$

is satisfied equally. Condition $F_{1\tau} + F_{2\tau} = N_\tau$ is used to specify required torque of traction wheel at definite resistance of driven one and required mode of vehicle motion along the specified lay of line. Static invariant two

$$\bar{F}_{1n} \cdot \left(\bar{N} \times \bar{r}_1\right) + \bar{F}_2 \cdot \left(\bar{N} \times \bar{r}_2\right) = 0$$

results in parallelism condition

$$\frac{F_{1n}}{F_{1b}} = \frac{F_{2n}}{F_{2b}} = \frac{N_n}{N_b}, \text{ and } \frac{l_1 F_{1n} - l_2 F_{2n}}{l_1 F_{1b} - l_2 F_{2b}} = \frac{N_n}{N_b}.$$

Thus, analytical solution determining equivalent contact of moving (control) forces for tandem-type two-wheel vehicle at various motion modes along spatial curved lay of line under the effect of gravity and aerodynamic forces is obtained. The closed vector dependences are shown in the form of quaternion matrices providing efficient computational algorithms.

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