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BRIEF COMMUNICATIONS

MARCHAUD'S INEQUALITY FOR MULTIPLE MODULES OF CONTINUITY IN METRIC SPACES

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For periodic functions of one variable in the metric spaces $L_\Psi[0,2\pi]$, we establish an analog of Marchaud's inequality for multiple modules of continuity.

Assume that, for real-valued functions $f(x),\ x \in R^1$, with period 1,

$$\Delta_t f(x) = f(x+t) - f(x), \qquad \Delta_t^k = \triangle_t\left(\Delta_t^{k-1}\right), \quad k \in \mathcal{N},$$

and

$$\omega_k(f,h)_X = \sup_{|t|\le h}\left\|\Delta_t^k f\right\|_X$$

is the module of continuity of order k in the space X.

In the case where $X = L_p,\ p \in [1,\infty]$, for $k < l,\ k, l \in \mathcal{N}$, parallel with the obvious inequality

$$\omega_l(f,h)_{L_p} \le 2^{l-k}\omega_k(f,h)_{L_p},$$

the following Marchaud inequality [1] is also true:

$$\omega_k(f,h)_{L_p} \le C_l h^k \int_h^1 \frac{\omega_l(f,s)_{L_p}}{s^k}\,\frac{ds}{s}, \tag{1}$$

where $h \in \left(0, \frac{1}{2}\right]$ and C_l is a positive constant independent of $p,\ h,$ and f.

For $p \in (1,\infty)$, Timan [2; 3, p. 41] proved sharper inequalities: For $h \in \left(0, \frac{1}{2}\right)$ and $k < l$,

$$C_{p,k} h^k \left(\int_h^1 \frac{\omega_l(f,s)_{L_p}^{\beta_1}}{s^{\beta_1 k}}\,\frac{ds}{s}\right)^{1/\beta_1} \le \omega_k(f,h)_{L_p} \le B_{p,k} h^k \left(\int_h^1 \frac{\omega_l(f,s)_{L_p}^{\beta_2}}{s^{\beta_2 k}}\,\frac{ds}{s}\right)^{1/\beta_2}, \tag{2}$$

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where

$$\beta_1 = \max(2, p), \qquad \beta_2 = \min(2, p), \qquad C_{p,k}, \qquad \text{and} \qquad B_{p,k} > 0.$$

Inequalities (2) were proved with the help of direct and inverse Jackson inequalities for the approximations of functions by trigonometric polynomials.

By using the same method, we prove an analog of the Marchaud inequality (1) in the metric spaces L_Ψ.

Let Ω be a set of functions $\Psi\colon R^1_+ \to R^1_+$, which are modules of continuity, i.e., Ψ is a continuous nondecreasing function, $\Psi(0) = 0$, and

$$\Psi(x + y) \le \Psi(x) + \Psi(y) \quad \text{for all} \quad x, y \in R^1_+.$$

Let L_0 be the set of measurable functions almost everywhere finite on a period. For $\Psi \in \Omega$, the set

$$L_\Psi = \left\{ f \in L_0\colon \|f\|_\Psi = \int_0^1 \Psi(|f(x)|)\, dx < \infty \right\}$$

is a linear metric space with the following metric:

$$\rho(f, g)_\Psi = \|f - g\|_\Psi.$$

Also let

$$M_\Psi(s) := \sup_{t>0} \frac{\Psi(st)}{\Psi(t)}, \quad s \in (0, \infty),$$

be the stretch function of Ψ [4] (Chap. II, Sec. 1).

We now prove the following theorem:

Theorem 1. *Suppose that $k, l \in \mathcal{N}$, $k < l$, and $M_\Psi\left(\frac{1}{2}\right) < 1$. Then, for all $h \in \left(0, \frac{1}{2}\right)$, the following inequality is true:*

$$\omega_k(f, h)_\Psi \le C \int_h^1 M_\Psi\left(\left(\frac{h}{s}\right)^k\right) \omega_l(f, s)_\Psi \frac{ds}{s}, \tag{3}$$

where C is a constant independent of f and h.

In the proof, we use the following results from the theory of approximation of functions in L_Ψ. Let

$$E_n(f)_\Psi = \inf_{\{c_k\}} \left\| f(x) - \sum_{k=-n}^{n} c_k e^{ik2\pi x} \right\|_\Psi$$

be the best approximation of f by trigonometric polynomials of degree at most n in L_Ψ.

Theorem A [5, 6]. *Suppose that* $\Psi \in \Omega,\ M_\Psi\left(\dfrac{1}{2}\right) < 1,$ *and* $k \in \mathcal{N}$. *Then*

$$\sup_n \sup_{f \in L_\Psi, f \neq \mathrm{const}} \frac{E_{n-1}(f)_\Psi}{\omega_k\left(f, \dfrac{1}{n}\right)_\Psi} < \infty, \tag{4}$$

$$\omega_k(f,h)_\Psi \le C \sum_{\nu=1}^{\left[\frac{1}{h}\right]} \frac{M_\Psi\big((\nu h)^k\big)}{\nu} E_{\nu-1}(f)_\Psi, \tag{5}$$

where $C = C(k,\Psi)$ *is a constant independent of* f *and* h.

Note that, in [7–9], these statements were proved for the spaces $L_p,\ p \in (0,1)$.

Proof of Theorem 1. For the sake of brevity, denote

$$\omega_k(h) := \omega_k(f,h)_\Psi, \qquad \omega_l(h) := \omega_l(f,h)_\Psi.$$

In what follows, all constants C_j depend only on $k,\ l,$ and Ψ.

To majorize $\omega_k\left(\dfrac{1}{2^n}\right)$, we successively apply inequalities (5) and (4) and obtain

$$\omega_k\left(\frac{1}{2^n}\right) \le C_1 \sum_{\nu=1}^{n+1} M_\Psi\left(2^{(\nu-n)k}\right) E_{2^{\nu-1}-1}(f)_\Psi$$

$$\le C_2 \sum_{\nu=1}^{n+1} M_\Psi(2^{(\nu-n)k})\, \omega_l\left(\frac{1}{2^{\nu-1}}\right).$$

Since the function M_Ψ is semimultiplicative [4] (Chap. II, Sec. 1), we get

$$M_\Psi\left(2^{(\nu-n)k}\right) = M_\Psi\left(2^{(\nu-1-n)k} 2^k\right)$$

$$\le M_\Psi\left(2^{(\nu-1-n)k}\right) M_\Psi\left(2^k\right) = C_3 M_\Psi\left(2^{(\nu-1-n)k}\right).$$

Further,

$$\omega_l\left(\frac{1}{2^{\nu-1}}\right) \le C_4 \omega_l\left(\frac{1}{2^\nu}\right).$$

It follows from (6) that

$$\omega_k\left(\frac{1}{2^n}\right) \le C_5 \sum_{\nu=1}^{n+1} M_\Psi\left(2^{(\nu-1)k} \frac{1}{2^{nk}}\right) \omega_l\left(\frac{1}{2^\nu}\right). \tag{6}$$

In view of the monotonicity of the functions M_Ψ and ω_l, we find

$$J_\nu := \int\limits_{\frac{1}{2^\nu}}^{\frac{1}{2^{\nu-1}}} M_\Psi\left(\left(\frac{1}{2^n}\frac{1}{s}\right)^k\right)\omega_l(s)\frac{ds}{s} \geq C_6\omega_l\left(\frac{1}{2^\nu}\right)M_\Psi\left(2^{(\nu-1)k}\frac{1}{2^{nk}}\right).$$

Thus, for $h = \dfrac{1}{2^n}$, we obtain inequality (3) from inequality (7):

$$\omega_k\left(\frac{1}{2^n}\right) \leq C_7\sum_{\nu=1}^{n+1} J_\nu = C_7\int\limits_{\frac{1}{2^{n+1}}}^{1} M_\Psi\left(\left(\frac{1}{2^n}\frac{1}{s}\right)^k\right)\omega_l(s)\,\frac{ds}{s}.$$

Further, for any h, we use standard reasoning. Let $h \in \left[\dfrac{1}{2^{n+1}}, \dfrac{1}{2^n}\right]$. In this case,

$$\begin{aligned}
\omega_k(h) &\leq \omega_k\left(\frac{1}{2^n}\right) \\
&\leq C_7\int\limits_{\frac{1}{2^{n+1}}}^{1} M_\Psi\left(\left(\frac{1}{2^n}\frac{1}{s}\right)^k\right)\omega_l(s)\,\frac{ds}{s} \\
&\leq C_7\int\limits_{\frac{h}{2}}^{1} M_\Psi\left(\left(2h\frac{1}{s}\right)^k\right)\omega_l(s)\frac{ds}{s} \\
&\leq C_7 M_\Psi(4^k)\int\limits_{h}^{1} M_\Psi\left(\left(\frac{h}{s}\right)^k\right)\omega_l(s)\,\frac{ds}{s}.
\end{aligned}$$

Theorem 1 is proved.

Remark 1. Inequality (3) is unimprovable in the following sense:

$$\sup_{h\in\left(0,\frac{1}{2}\right)}\ \sup_{f\in L_\Psi, f\neq\mathrm{const}} \frac{\omega_k(f,h)_\Psi}{\int_h^1 M_\Psi\left(\left(\dfrac{h}{s}\right)^k\right)\omega_l(f,s)_\Psi\,\dfrac{ds}{s}} > 0. \tag{7}$$

For a given $h \in \left(0, \dfrac{1}{l}\right)$, let $f(x) = \chi_{[0,h]}(x)$ be the characteristic function of the segment $[0,h]$ and let $s > h$. Then

$$\omega_k(f,h)_\Psi = \left\|\Delta_h^k f\right\|_\Psi = \left\|\sum_{j=0}^{k}(-1)^{k-j}C_k^j f(x+jh)\right\|_\Psi = \sum_{j=0}^{k}\Psi\left(C_k^j\right)h,$$

$$\omega_l(f,s)_\Psi \le \sum_{\nu=0}^{l} \|C_l^\nu f(x+s)\|_\Psi = \sum_{\nu=0}^{l} \Psi\left(C_l^\nu\right) h,$$

$$\int_h^1 M_\Psi\left(\left(\frac{h}{s}\right)^k\right) \omega_l(f,s)_\Psi \frac{ds}{s} \le \sum_{\nu=0}^{l} \Psi\left(C_l^\nu\right) h \frac{1}{k} \int_{h^k}^1 M_\Psi(t) \frac{dt}{t}.$$

This yields (8) if

$$\int_0^1 M_\Psi(t) \frac{dt}{t} < \infty.$$

It is known [4] (Chap. II, Sec. 1) that, for the stretch function M_Ψ, there exists a lower stretch index Υ_Ψ such that, for any $\varepsilon > 0$, one can find a constant $C_\varepsilon > 0$ such that, for all $t \in (0,1]$, the inequalities

$$t^{\Upsilon_\Psi} \le M_\Psi(t) \le C_\varepsilon t^{\Upsilon_\Psi - \varepsilon}$$

are true. If $\Psi \in \Omega$, then $\Upsilon_\Psi \in [0,1]$. However, it follows from the condition $M_\Psi\left(\frac{1}{2}\right) < 1$ that

$$\Upsilon_\Psi > 0.$$

Therefore, for sufficiently small ε, we get

$$\int_0^1 M_\Psi(t) \frac{dt}{t} \le C_\varepsilon \int_0^1 t^{\gamma_\Psi - \varepsilon} \frac{dt}{t} < \infty.$$

Remark 2. If $\int_0^1 M_\Psi\left(t^k\right) M_{\omega_l}\left(\frac{1}{t}\right) \frac{dt}{t} < \infty$, then $\omega_k(f,h)_\Psi \asymp \omega_l(f,h)_\Psi$.

Indeed, since

$$\omega_l(f,s)_\Psi = \omega_l\left(f, h\frac{s}{h}\right)_\psi \le \omega_l(f,h)_\Psi M_{\omega_l}\left(\frac{s}{h}\right)_\Psi,$$

we find

$$\frac{\omega_k(f,h)_\Psi}{\omega_l(f,h)_\Psi} \le C \int_h^1 M_\Psi\left(\left(\frac{h}{s}\right)^k\right) M_{\omega_l}\left(\frac{s}{h}\right) \frac{ds}{s} = C \int_h^1 M_\Psi(t^k) M_{\omega_l}\left(\frac{1}{t}\right) \frac{dt}{t}.$$

In particular, if $M_{\omega_l}(y) \le K y^{\delta_l}$ for $y \ge 1$, then the condition $\delta_l < k\Upsilon_\Psi$ implies that

$$\omega_k(f,h)_\Psi \asymp \omega_l(f,h)_\Psi.$$

This statement was proved in [5].

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