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BRIEF COMMUNICATIONS

SHARP ESTIMATES FOR THE BEST APPROXIMATIONS OF SMOOTH FUNCTIONS IN $C_{2\pi}$ IN TERMS OF LINEAR COMBINATIONS OF THE MODULES OF CONTINUITY OF THEIR DERIVATIVES

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For the best approximations of $e_{n-1}(f)$ functions in $C^1_{2\pi}$ by trigonometric polynomials, V. Zhuk proved the exact Jackson's inequality $e_{n-1}(f) \leq \frac{\pi}{4n}\omega\Big(f', \frac{\pi}{n}\Big)$. We establish the following version of exact Jackson's inequality: $e_{n-1}(f) \leq \frac{\pi}{4n}\left(\frac{1}{2}\omega\Big(f', \frac{\pi}{2n}\Big) + \frac{1}{2}\omega\Big(f', \frac{\pi}{n}\Big)\right)$.

Let $C[-\pi, \pi]$ be the space of real-valued continuous 2π-periodic functions with the norm

$$\|f\| = \max\{|f(x)|; x \in R^1\},$$

let

$$e_{n-1}(f) = \inf_{\{c_k\}}\left\{\left\|f(x) - \sum_{k=-(n-1)}^{n-1} c_k e^{ikx}\right\|;\ c_{-k} = \bar{c}_k\right\}$$

be the best approximation of f by a subspace of trigonometric polynomials $\{T_{n-1}\}$ of degree at most $n-1$, $n \in \mathbb{N}$, and let

$$\omega(f, h) = \sup\{\|\Delta_t f\|; |t| \leq h\}, \qquad \Delta_t f(x) = f\left(x + \frac{t}{2}\right) - f\left(x - \frac{t}{2}\right)$$

be the module of continuity of f.

In order to find the upper bound for the quantity $e_{n-1}(f)$ via the values of the module of continuity of f, it is customary to use the Jackson–Korneichuk inequality {see [1, 2] (Sec. 9.2)}

$$e_{n-1}(f) \leq \omega\Big(f, \frac{\pi}{n}\Big).$$

For all $n \in \mathbb{N}$, this inequality is unimprovable. More exactly, for all $n \in \mathbb{N}$, we have

$$1 - \frac{1}{2n} \leq \sup_{\substack{f \in C[-\pi,\pi] \\ f \neq \text{const}}} \frac{e_{n-1}(f)}{\omega\Big(f, \frac{\pi}{n}\Big)} \leq 1.$$

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