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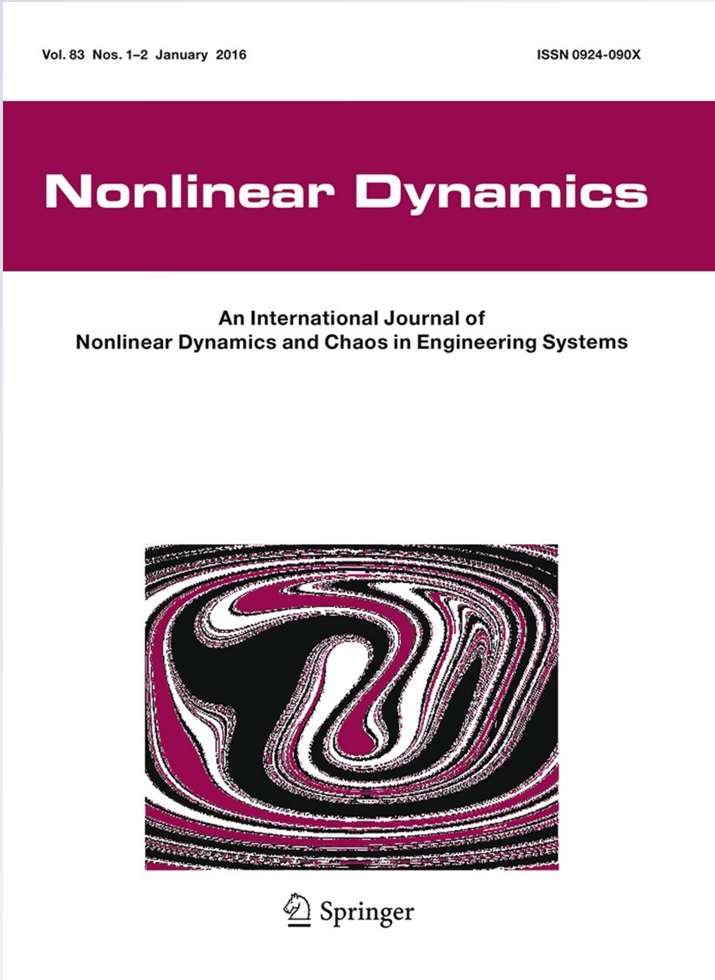

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ORIGINAL PAPER

Role of logistic and Ricker's maps in appearance of chaos in autonomous quadratic dynamical systems

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Abstract New existence conditions of a chaotic behavior for wide class of $(n + 1)$-dimensional autonomous quadratic dynamical systems are suggested. It is shown that in all such systems the chaotic dynamics is generated by 1D discrete map by some combination of the logistic map $f(x) = \lambda x(1-x)$; $\lambda > 0$ and Ricker's map $g(x) = x \exp(\mu - x)$; $\mu > 0$.

Keywords Ordinary autonomous quadratic differential equations system · Limit cycle · Saddle focus · 1D discrete map · Chaos

1 Introduction

Chaos is a very interesting nonlinear phenomenon, which has been intensively studied for last four decades. Many potential applications have come true in secure communication, laser and biological systems, and other areas (see, for example, Refs. [1–3] and many references cited therein).

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From the mathematical point of view, there are two basic methods of search of chaotic systems. They are based either on establishment of the existence fact of a homoclinic orbit in a given dynamical system or on construction for the given system of a discrete map and proof of its chaotic.

There is a huge number of papers devoted to the search of homoclinic orbits in 3D systems of differential equations (see, for example, [4–13]). It is necessary also to mention the series of publications [14–19], in which new chaotic systems were created as a result of generalization of the classic Lorenz system. (It generalization as the final result to the search of homoclinic orbits was reduced).

The construction of discrete maps for continuous dynamical systems is still small studied. Here basic results are contained, for example, in [20–28]. The main idea of these papers is that properties of the being created discrete maps, which describe a behavior of continuous dynamical systems, are based on the well-known properties of the Ricker map $f(x) = x\exp(r-x)$ or the logistic map $g(x) = rx(1-x)$. Our approach to research of chaos in $(n + 1)$-dimensional autonomous quadratic systems is also based on this idea.

The most general approach at the study of chaos in the continuous $(n+1)$-dimensional system consists in finding of a basin of attraction for this system. The simplest situation arises up then, when the basin of attraction is whole space $\mathbb{R}^{n+1}$. In other words, for existence of the basin it is sufficiently that all solutions of the system were bounded at any initial data. Exactly this approach will be realized in the present work. In the total a significant part of results about existence of chaotic dynamics obtained in [24,25] for 3D quadratic systems it will be carried on $(n+1)$-dimensional case, where $n > 1$.

Finally, we notice that all systems considered further are generalizations of the following 3D system ([20]):

$$\begin{cases} \dot{x}(t) = a_{11}x(t) + a_{12}y(t) + a_{13}z(t) + h_{11}y^2(t) + h_{12}y(t)z(t) + h_{22}z^2(t), \\ \dot{y}(t) = a_{21}x(t) + a_{22}y(t) + a_{23}z(t) + x(t)(b_1 y(t) + b_2 z(t)), \\ \dot{z}(t) = a_{31}x(t) + a_{32}y(t) + a_{33}z(t) + x(t)(c_1 y(t) + c_2 z(t)). \end{cases} \tag{1}$$

2 Bounded solutions of quadratic dynamical systems

Consider the following $(n+1)$-dimensional autonomous quadratic system of differential equations

$$\dot{\mathbf{x}}(t) = A\mathbf{x} + x_0 B\mathbf{x} + \mathbf{f}(\mathbf{x}),\ \mathbf{x} = (x_0, x_1, \ldots, x_n)^T \in \mathbb{R}^{n+1}, \quad n \geq 1. \tag{2}$$

Here

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

$$B = \begin{pmatrix} 0 & b_{01} & \cdots & b_{0n} \\ 0 & b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n1} & \cdots & b_{nn} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

$$\mathbf{f}(\mathbf{x}) = (f_0(x_1, \ldots, x_n),\ f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))^T \in \mathbb{R}^{n+1},$$

and

$$f_0(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} c_{ij}^{(0)} x_i x_j,\ f_1(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} c_{ij}^{(1)} x_i x_j, \ldots, f_n(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} c_{ij}^{(n)} x_i x_j$$

are real quadratic forms.

Suppose that the matrix B has rank n. Then, as is shown in [24,25], by suitable real linear nonsingular transformation $S = (s_{ij}) \in \mathbb{R}^{(n+1)\times(n+1)}$ of variables $x_0, x_1, \ldots, x_n$ of the type

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0n} \\ 0 & s_{11} & \cdots & s_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & s_{n1} & \cdots & s_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix},$$

system (2) can be reduced to the same kind (2), but with other coefficients; $i, j = 0, 1, \ldots, n$. In addition, in this new presentation the matrix B will have the form

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & b_{11} & \cdots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n1} & \cdots & b_{nn} \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}. \tag{3}$$

(With the purpose of simplification of further exposition, we left in the new system the designations accepted in system (2). Besides, under the conditions $n = 2$, $f_1(\mathbf{x}) = f_2(\mathbf{x}) \equiv 0$, and condition (3) system (2) coincides with system (1).)

Introduce in system (2) new real variables $\rho > 0, \phi_1, \ldots, \phi_n$ under the formulas: $x_0 = x$, $x_1 = \rho\cos\phi_1, \ldots, x_n = \rho\cos\phi_n$, where $\cos^2\phi_1 + \cdots + \cos^2\phi_n \equiv 1$. Then, after replacement of variables and multiplication of the second, third,…, and the last equations of system (2) on the corresponding coordinates of row vector $(\cos\phi_1, \ldots, \cos\phi_n)$ and summation, we get the first and second equations of system (2) in such aspect

$$\begin{cases} \dot{x}(t) = gx + g_1(\phi_1, \ldots, \phi_n)\rho + g_{22}(\phi_1, \ldots, \phi_n)\rho^2, \\ \dot{\rho}(t) = h(\phi_1, \ldots, \phi_n)x + h_1(\phi_1, \ldots, \phi_n)\rho \\ \qquad + h_{12}(\phi_1, \ldots, \phi_n)x\rho + h_{22}(\phi_1, \ldots, \phi_n)\rho^2, \end{cases} \tag{4}$$

where $\phi_i = \phi_i(t)$, $i = 1, \ldots, n$, and

$$g = a_{00},$$

$$g_1(\phi_1, \ldots, \phi_n) = \sum_{j=1}^{n} a_{0i}\cos\phi_i,$$

$$g_{22}(\phi_1, \ldots, \phi_n) = \sum_{i,j=1}^{n} c_{ij}^{(0)}\cos\phi_i\cos\phi_j,$$

$$h(\phi_1,\ldots,\phi_n)=\sum_{i=1}^{n}a_{i0}\cos\phi_i,$$

$$h_1(\phi_1,\ldots,\phi_n)=\sum_{i=1}^{n}(\cos\phi_i)\cdot\Big(\sum_{j=1}^{n}a_{ij}\cos\phi_j\Big),$$

$$h_{12}(\phi_1,\ldots,\phi_n)=\sum_{i=1}^{n}(\cos\phi_i)\cdot\Big(\sum_{j=1}^{n}b_{ij}\cos\phi_j\Big),$$

$$h_{22}(\phi_1,\ldots,\phi_n)=\sum_{i=1}^{n}(\cos\phi_i)\cdot f_i(\cos\phi_1,\ldots,\cos\phi_n).$$

Let $u_1=\cos\phi_1,\ldots,u_n=\cos\phi_n$, where $u_1^2+\cdots+u_n^2=1$ (thus, we have $x_1=\rho u_1,\ldots,x_n=\rho u_n$). Introduce the quadratic forms of degree 2: $g_{22}(u_1,\ldots,u_n)$, $h_{12}(u_1,\ldots,u_n)$, and $\Delta(u_1,\ldots,u_n)\equiv g\cdot h_1(u_1,\ldots,u_n)-g_1(u_1,\ldots,u_n)\cdot h(u_1,\ldots,u_n)$. Besides, we also introduce the polynomial

$$D(u_1,\ldots,u_n)\equiv h_{22}^2(u_1,\ldots,u_n)+4g_{22}(u_1,\ldots,u_n)\cdot h_{12}(u_1,\ldots,u_n)$$

of degree 6.

From the mathematical analysis, it is well known that any real continuous function $w(u_1,\ldots,u_n)$ reach on the sphere $\mathbb{S}:=u_1^2+\cdots+u_n^2=1$ of it the greatest and the least values.

Let $g_{1G}\geq 0$, g_{22G}, h_{1G}, h_{12G}, and $h_{22G}\geq 0$ be the greatest values on the sphere $\mathbb{S}$ of the functions $g_1(u_1,\ldots,u_n)$, $g_{22}(u_1,\ldots,u_n)$, $h_1(u_1,\ldots,u_n)$, $h_{12}(u_1,\ldots,u_n)$, and $h_{22}(u_1,\ldots,u_n)$, respectively.

Consider the system

$$\begin{cases}\dot{x}_a(t)=gx_a+g_{1G}\rho_a+g_{22G}\rho_a^2,\\ \dot{\rho}_a(t)=h(u_1,\ldots,u_n)x_a+h_{1G}\rho_a+h_{12G}x_a\rho_a\\ \qquad\quad +h_{22G}\rho_a^2,\end{cases}\tag{5}$$

where $h(u_1,\ldots,u_n)$ is a real bounded function.

It is clear that for systems (4) and (5), we have $x(t)\leq x_a(t)$ and $\rho(t)\leq\rho_a(t)$. Thus, according to comparison principle, from boundedness of solutions of system (5) it follows that solutions $x(t)$ and $\rho(t)$ of system (4) are also bounded.

Use the following theorem, which is generalization of the known Theorem 1 [29].

Theorem 1 *The quadratic system (5) has all of its trajectories bounded for* $t\geq 0$ *if and only if the conditions* $g<0$, $g_{22G}>0$, $h_{12G}<0$, *and* $h_{22G}^2+4g_{22G}\cdot h_{12G}<0$ *are valid.*

It is clear that if the quadratic form $g_{22}(u_1,\ldots,u_n)$ is positive definite and the quadratic form $h_{12}(u_1,\ldots,u_n)$ is negative definite (or vice versa), then Theorem 1 can be applied to system (4). (In this case, for all real numbers $u_1,\ldots,u_n$ such that the condition $u_1^2+\cdots+u_n^2=1$ is satisfied, we have $g_{22}(u_1,\ldots,u_n)\cdot h_{12}(u_1,\ldots,u_n)\neq 0$ and the greatest value of the function $h_{12}(u_1,\ldots,u_n)$ is negative.)

Theorem 2 *Assume that for system (2) the matrix B has form (3). If* $a_{00}<0$ *and the magnitude*

$$\max_{(u_1,\ldots,u_n)\in\mathbb{S}}D(u_1,\ldots,u_n)<0,\tag{6}$$

then all trajectories of system (2) are bounded for $t\geq 0$.

Proof It is clear that if $\forall(u_1,\ldots,u_n)\in\mathbb{S}$, then we have $D(u_1,\ldots,u_n)<0$. Therefore, from here it follows that $\forall(u_1,\ldots,u_n)\in\mathbb{S}$ $g_{22}(u_1,\ldots,u_n)\cdot h_{12}(u_1,\ldots,u_n)<0$. □

Now suppose that in system (5) we have $g_{22G}<0$ and $h_{12G}>0$. Then the replacement of variable $x_a\to-\overline{x}_a$ lead to new system (5) in which we will have $g_{22G}>0$ and $h_{12G}<0$. Since at this replacement of variable x_a the inequalities $g<0$ and $x_1^2(t)+\cdots+x_n^2(t)=\rho^2(t)(\cos^2\phi_1(t)+\cdots+\cos^2\phi_n(t))\leq\rho_a^2(t)$ are saved, then the application of Theorem 1 completes the proof of Theorem 2. □

In further reasonings the following theorem plays a key role.

Theorem 3 *Assume that for system (2) we have* $a_{00}<0$, $a_{10}=\cdots=a_{n0}=0$, *and the quadratic form* $h_1(u_1,\ldots,u_n)$ *is positive definite. Then under the conditions of Theorem 2 in this system there exists a unique stable limit cycle.*

Proof It is clear that in system (5) we can write $h(u_1,\ldots,u_n)\equiv 0$, $h_{1G}>0$, $h_{12G}<0$, $h_{22G}\geq 0$, $g_{1G}\geq 0$, and $g_{22G}>0$. Then instead of system (5), we get the following system

$$\begin{cases}\dot{x}_b(t)=gx_b+g_{1G}\rho_b+g_{22G}\rho_b^2,\\ \dot{\rho}_b(t)=h_{1G}\rho_b+h_{12G}x_b\rho_b+h_{22G}\rho_b^2,\end{cases}\tag{7}$$

where we again have $x_a(t)\leq x_b(t)$ and $\rho_a(t)\leq\rho_b(t)$.

Now introduce in system (7) new real variables $\rho_c=-\rho_b>0$ and $x_c=x_b$. Then system (7) transforms in

$$\begin{cases}\dot{x}_c(t)=gx_c-g_{1G}\rho_c+g_{22G}\rho_c^2,\\ \dot{\rho}_c(t)=h_{1G}\rho_c+h_{12G}x_c\rho_c-h_{22G}\rho_c^2.\end{cases}\tag{8}$$

Thus, if instead of (8) we will consider the following system

$$\begin{cases} \dot{x}_d(t) = gx_d + g_{22G}\rho_d^2, \\ \dot{\rho}_d(t) = h_{1G}\rho_d + h_{12G}x_d\rho_d. \end{cases} \tag{9}$$

Then from here it follows that $x_c(t) \le x_d(t)$ and $\rho_c(t) \le \rho_d(t)$.

Finally, if in system (9) we put $\rho_d = -\rho_e$ and $x_d = x_e$, then we get the same system (9), but with the condition $\rho_e > 0$.

Thus, at the initial values $x_{d0} > 0$ and $\rho_{d0} > 0$ system (9) coincides with system (17) [23]. In the same paper [23] (Theorem 5 and its Corollary), it was shown that the system (17) [23] had a unique stable limit cycle. Consequently, according to Comparison Principle, system (7) (and systems (5) and (2)) also must have a unique stable limit cycle. □

It is difficult to check up condition (6). Therefore, in applications the following obvious corollary of Theorem 2 can be useful.

Theorem 4 *Assume that for system (2) the matrix B has form (3) and $a_{00} < 0$. Suppose also that $f_1(x_1,\dots,x_n) = .. = f_n(x_1,\dots,x_n) \equiv 0$. If the quadratic form $I_1(x_1,\dots,x_n) = \rho^2 g_{22}(u_1,\dots,u_n)$ is positive definite and the quadratic form $I_2(x_1,\dots,x_n) = \rho^2 h_{12}(u_1,\dots,u_n)$ is negative definite (or vice versa), then all trajectories of system (2) are bounded for $t \ge 0$*

3 Appearance of chaotic solutions in system (2)

A basic result, which will be proved in this section, contains in the following theorem.

Theorem 5 *Suppose that for system (4) the following conditions:*

(i) $g < 0$;

(ii) the quadratic form $\Delta(u_1,\dots,u_n)$ is negative definite;

(iii) for any nonzero vector $(u_1,\dots,u_n) \in \mathbb{R}^n$ the function $D(u_1,\dots,u_n) < 0$;

(iv) $\liminf_{t\to\infty} \rho(t) = \liminf_{t\to\infty} \sqrt{x_1^2(t) + \dots + x_n^2(t)} = 0$

(from this condition it follows that $\forall\epsilon > 0$ there exists a numerical sequence $t_m \to \infty$ as $m \to \infty$ such that $\rho(t_m) < \epsilon$) are fulfilled.

Then in system (2) there is a chaotic dynamics.

Corollary *Under the conditions of Theorem 5 the chaotic behavior of solutions of system (2) is generated by*

either 1D iterated process

$$v_{k+1} = v_k \exp(\lambda + \nu v_k - \mu v_k^2),\ v_k > 0;$$
$$k = 0, 1, 2, \dots; \lambda > 0, \mu > 0, \nu \in \mathbb{R}, \tag{10}$$

if the condition $h_{22}(u_1,\dots,u_n) \not\equiv 0$ is valid

or 1D iterated process

$$v_{k+1} = v_k(1 - v_k)\exp(\lambda_1 + \nu_1 v_k - \mu_1 v_k^2),\ v_k \in [0, 1];$$
$$k = 0, 1, 2, \dots; \lambda_1 > 0, \mu_1 > 0, \nu_1 \in \mathbb{R}, \tag{11}$$

if the condition $h_{22}(u_1,\dots,u_n) \equiv 0$ is valid.

Proof It is obvious that from conditions (i)–(iii) follows the boundedness of all solutions of system (2). Nevertheless, precisely the conditions of Theorem 2 will allow to construct process (10), which generates chaos in system (2). □

(d1) Indeed, from condition (iii) it follows that $\lim_{t\to\infty} \rho(t) < \infty$. (Otherwise would be $\lim_{m\to\infty} t_m < \infty$.)

Suppose opposite: there exists a point t_s^* (it can be $t_s^* = \infty$) such that $\lim_{t\to t_s^*} x(t) = \lim_{t\to t_s^*} \rho(t) = \infty$, $\rho(t) > 0$, and $x(t) > 0$. Then using L'Hospital's rule for system (4), we get

$$\begin{aligned} \lim_{t\to t_s^*} \frac{x(t)}{\rho(t)} &= \frac{\infty}{\infty} = \lim_{t\to t_s^*} \frac{\dot{x}(t)}{\dot{\rho}(t)} \\ &= \lim_{t\to t_s^*} \frac{gx + g_1(\phi_1,\dots,\phi_n)\rho + g_{22}(\phi_1,\dots,\phi_n)\rho^2}{h(\phi_1,\dots,\phi_n)x + h_1(\phi_1,\dots,\phi_n)\rho + h_{12}(\phi_1,\dots,\phi_n)x\rho + h_{22}(\phi_1,\dots,\phi_n)\rho^2} \\ &= \lim_{t\to t_s^*} \frac{g_{22}(\phi_1,\dots,\phi_n)}{h_{12}(\phi_1,\dots,\phi_n)(x/\rho) + h_{22}(\phi_1,\dots,\phi_n)}. \end{aligned}$$

From here it follows that

$$h_{12} \lim_{t\to t_s^*} \frac{x^2(t)}{\rho^2(t)} + h_{22} \lim_{t\to t_s^*} \frac{x(t)}{\rho(t)} - g_{22} = 0. \tag{12}$$

A discriminant of the last quadratic equation can be calculated on the formula

$$D(u_1,\ldots,u_n) = h_{22}^2(u_1,\ldots,u_n) + 4h_{12}(u_1,\ldots,u_n)g_{22}(u_1,\ldots,u_n).$$

Since $D(u_1,\ldots,u_n) < 0$, then equation (12) does not have the solutions. A contradiction with the suppositions $\rho(t) \to \infty$ and $x(t) \to \infty$ was got. Thus, $\forall t > 0$ we have $\rho(t) < \infty$ and $x(t) < \infty$.

(d2) Now taking into account Theorem 3 we can consider that for some values of parameters, system (2) has a periodic solution. (It means that system (4) has also the periodic solution.) Suppose also that $\phi_i(t_k) = \phi_i(t_0)+T\cdot k$, where $t_0 \geq 0$, $T \leq N\cdot\pi$, and N is positive integer; $k = 0, 1, 2, \ldots$; $i = 1, \ldots, n$. Introduce the designations:

$$\begin{aligned}
&g = a_{00} = \xi_{11} < 0\\
&g_1(\cos\phi_1(t_k),\ldots,\cos\phi_n(t_k)) = \xi_{12} = \text{const},\\
&g_{22}(\cos\phi_1(t_k),\ldots,\cos\phi_n(t_k)) = \zeta_{22} = \text{const},\\
&h(\cos\phi_1(t_k),\ldots,\cos\phi_n(t_k)) = \xi_{21} = \text{const},\\
&h_1(\cos\phi_1(t_k),\ldots,\cos\phi_n(t_k)) = \xi_{22} = \text{const},\\
&h_{12}(\cos\phi_1(t_k),\ldots,\cos\phi_n(t_k)) = \eta_{12} = \text{const},\\
&h_{22}(\cos\phi_1(t_k),\ldots,\cos\phi_n(t_k)) = \eta_{22} = \text{const}.
\end{aligned}$$

Consider the infinite sequence of systems of differential equations

$$\begin{cases} \dot{x}_k(t) = \xi_{11}x_k + \xi_{12}\rho_k + \zeta_{22}\rho_k^2,\\ \dot{\rho}_k(t) = \xi_{21}x_k + \xi_{22}\rho_k + \eta_{12}x_k\rho_k + \eta_{22}\rho_k^2 \end{cases} \quad (13)$$

instead of system (4). (Here each of systems (13) is considering in a small neighborhood $\mathbb{O}_k$ of the point t_k: $t \in \mathbb{O}_k$, $k = 0, 1, 2, \ldots$. As initial conditions x_{k0}, ρ_{k0} for each of systems (13) the solutions of system (4) in the point t_k are appointed.)

Suppose that the time t_0 also satisfies to the condition

$$\dot{x}(t_0) = \xi_{11}x_0 + \xi_{12}\rho_0 + \zeta_{22}\rho_0^2 = 0.$$

By virtue of periodicity of solutions of system (4), we can construct the sequence t_0, t_1,…, t_k,…such that for the first equation of system (13) the condition $\xi_{11}x_k + \xi_{12}\rho_k + \zeta_{22}\rho_k^2 = 0$ will be fulfilled $\forall t_k$, $k = 0, 1, 2, \ldots$. From here it follows that

$$x_k = -\frac{\xi_{12}\rho_k + \zeta_{22}\rho_k^2}{\xi_{11}};\ k = 0, 1, 2, \ldots. \quad (14)$$

Assume that $h(\cos\phi_1,\ldots,\cos\phi_n) \equiv 0$ and $h_{22}(\cos\phi_1,\ldots,\cos\phi_n) \not\equiv 0$. (According to Theorem 3 system (4) can have a periodic solution.) Then from the second equation of system (4), it follows that

$$\begin{aligned}\dot{\rho}(t) = {}& h_{22}(\cos\phi_1,\ldots,\cos\phi_n)\rho^2(t)\\ &+ [h_{12}(\cos\phi_1,\ldots,\cos\phi_n)x(t)\\ &+ h_1(\cos\phi_1,\ldots,\cos\phi_n)]\rho(t). \end{aligned} \quad (15)$$

The solution of equation (15) may be derived under the formula

$$\rho(t) = \frac{\rho_0 \exp(q(t))}{1-\rho_0\int_{t_0}^{t} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau}, \quad (16)$$

where

$$\begin{aligned} q(t) = \int_{t_0}^{t} & [h_{12}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))x(\tau)\\ &+ h_1(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))]d\tau \end{aligned}$$

and $\forall t > t_0 \int\limits_{t_0}^{t} \exp(q(\tau))d\tau > 0$.

From formulas (15) and (16) we have

$$\rho_{k+1} = \rho_k \exp(q(t_{k+1}) - q(t_k)) \frac{1-\rho_0\int_{t_0}^{t_k} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau}{1-\rho_0\int_{t_0}^{t_{k+1}} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau},$$

where

$$\begin{aligned} q(t_{k+1})-q(t_k) = \int\limits_{t_k}^{t_{k+1}} & [h_{12}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))x(\tau)\\ &+ h_1(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))]d\tau. \end{aligned}$$

We transform this formula taking account of formula (14). Then we derive

$$q(t_{k+1}) - q(t_k) = \\ -\frac{\rho_k^2}{g}\int\limits_{t_k}^{t_{k+1}} g_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))h_{12}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau - \\ -\frac{\rho_k}{g}\int_{t_k}^{t_{k+1}} h_{12}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))g_1(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau + \\ \int_{t_k}^{t_{k+1}} h_1(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau = -E\rho_k^2 + F\rho_k + G,$$

where

$$E = (1/g)\int\limits_{t_k}^{t_{k+1}} h_{12}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))g_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau > 0,$$

$$F = -(1/g)\int\limits_{t_k}^{t_{k+1}} h_{12}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))g_1(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau,$$

and by virtue of the conditions (i) – (iii)

$$G = (1/g)\int\limits_{t_k}^{t_{k+1}} gh_1(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau \\ \equiv (1/g)\int\limits_{t_k}^{t_{k+1}} \Delta(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))d\tau > 0.$$

In addition, we construct the function

$$\Theta = \lim_{k\to\infty}\frac{1-\rho_0\int\limits_{t_0}^{t_k} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau}{1-\rho_0\int\limits_{t_0}^{t_{k+1}} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau} \\ = 1 + \lim_{k\to\infty}\frac{\rho_0\int\limits_{t_k}^{t_{k+1}} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau}{1-\rho_0\int\limits_{t_0}^{t_{k+1}} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau} \\ = 1 + \lim_{k\to\infty}\frac{\rho_0 h_{22}(\cos\phi_1(t^*),\ldots,\cos\phi_n(t^*))\int\limits_{t_k}^{t_{k+1}}\exp(q(\tau))d\tau}{1-\rho_0\int\limits_{t_0}^{t_{k+1}} h_{22}(\cos\phi_1(\tau),\ldots,\cos\phi_n(\tau))\exp(q(\tau))d\tau},$$

where $t^* \in (t_k, t_{k+1}), q(\tau) < p\tau,\ p < 0$. From here it follows that $\lim\limits_{k\to\infty}\int\limits_{t_k}^{t_{k+1}}\exp(q(\tau))d\tau = 0$. Thus, we have $\Theta = 1$.

Finally, $\forall \rho_k > 0$ and $k \to \infty$, we obtain

$$\rho_{k+1} = \rho_k \exp\Big(-E\rho_k^2 + F\rho_k + G\Big);\ k = 0, 1, 2, \ldots, \tag{17}$$

where $E > 0$, $G > 0$. Then from here the discrete process (10) may be derived. The state of chaos of the map (17) on the interval $[0, \infty)$ was proved in [22].

(d3) Now let $h(\cos\phi_1, \ldots, \cos\phi_n) \not\equiv 0$. We again take advantage of formula (14). Let $t \in (t_k - \delta, t_k + \delta)$, where $\delta \neq 0$ is enough small. Then from the second equation of system (4) it follows that

$$\begin{aligned}\dot{\rho}(t) = \frac{\rho(t)}{g}\Big[&\Delta(\cos\phi_1(t), \ldots, \cos\phi_n(t)) + (gh_{22}(\cos\phi_1(t), \ldots, \cos\phi_n(t))\\&- h(\cos\phi_1(t), \ldots, \cos\phi_n(t))g_{22}(\cos\phi_1(t), \ldots, \cos\phi_n(t))\\&- g_1(\cos\phi_1(t), \ldots, \cos\phi_n(t))h_{12}(\cos\phi_1(t), \ldots, \cos\phi_n(t)))\rho(t)\\&- h_{12}(\cos\phi_1(t), \ldots, \cos\phi_n(t))g_{22}(\cos\phi_1(t), \ldots, \cos\phi_n(t))\rho^2(t)\Big].\end{aligned}$$

The solution of this equation can be represented in the integral form

$$\begin{aligned}\rho(t) = \rho(t_0)\exp\Bigg[\frac{1}{g}\int_{t_0}^{t}\Bigg[&\Delta(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\\&+ (gh_{22}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\\&- h(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))g_{22}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\\&- g_1(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))h_{12}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau)))\rho(\tau)\\&- h_{12}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))g_{22}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\rho^2(\tau)\Bigg]d\tau\Bigg].\end{aligned}$$

Suppose that in last formula the variable t takes two values: t_k and t_{k+1}, $t_k < t_{k+1}$. Then we can define the numbers $\rho(t_k) = \rho_k > 0$, $\rho(t_{k+1}) = \rho_{k+1} > 0$. In this case we can rewrite the formula for $\rho(t)$ as

$$\begin{aligned}\rho_{k+1} = \rho_k\exp\Bigg[\frac{1}{g}\int_{t_k}^{t_{k+1}}\Bigg[&\Delta(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\\&+ (gh_{22}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\\&- h(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))g_{22}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\\&- g_1(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))h_{12}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau)))\rho(\tau)\\&- h_{12}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))g_{22}(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))\rho^2(\tau)\Bigg]d\tau\Bigg].\end{aligned}$$

Taking into account the known first theorem about mean value, we can derive the formula $\rho_{k+1} = \rho_k\exp\left(-E\rho_k^2 + F\rho_k + G\right)$; $k = 0, 1, 2, \ldots$, in which $E > 0$ and $G > 0$ are the same that in (17);

$$F = (1/g)\int_{t_k}^{t_{k+1}}\Big[-h_{12}(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))g_1(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau)) - h(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))g_{22}(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau)) + h_{22}(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))g(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))\Big]d\tau.$$

The state of chaos of the map $f(\rho) = \rho\cdot\exp(G + F\rho - E\rho^2)$ on the interval $[0, \infty)$ can be proved by the methods offered in [22].

Indeed, consider the exponential map

$$v_{k+1} = \Psi(v_k, \lambda, \nu, \mu) \equiv v_k \exp(\lambda + \nu v_k - \mu v_k^2),$$
$$v_k > 0;\ k = 0, 1, 2, \dots; \lambda > 0, \mu > 0, \nu \in \mathbb{R}.$$

Let v_k^* be the minimal fixed point of mapping $\Psi^{(k)}(v, \lambda, \nu, \mu)$. It is known that for some $\lambda = \lambda^*, \nu = \nu^*, \mu = \mu^*$ the map Ψ is chaotic and $\lim_{k\to\infty} v_k^*(\lambda^*, \nu^*, \mu^*) = 0$. Then from the condition (iv) of Theorem 5 it follows that at the parameters $\lambda = \lambda^*, \nu = \nu^*, \mu = \mu^*$ process (10) generates the subsequence $v_{m_1},\dots, v_{m_k},\dots$, for which $v_{m_k} = \lim_{t\to t^*_{m_k}} v(t^*_{m_k}) < \epsilon \approx 0$, $k \geq 1$. It means that in system (2) there is a chaotic dynamics.

Thus, the conclusions of all items (d1) – (d3) allow to complete the proof of Theorem 5 and formula (10) of its Corollary.

(d4) Now let $h_{22}(\cos\phi_1,\dots,\cos\phi_n) \equiv 0$. Then the solution of the second equation of system (4) can be represented in the integral form

$$\rho(t) = \exp\left[\int_0^t (h_1(\cos\phi_1(\omega),\dots,\cos\phi_n(\omega)) + h_{12}(\cos\phi_1(\omega),\dots,\cos\phi_n(\omega))x(\omega))d\omega\right] \times \left[\rho_0 + \int_0^t (h(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))x(\tau) \times \exp\left[-\int_0^\tau (h_1(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau)) + h_{12}(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))x(\tau))d\tau\right] d\tau\right], t > \tau, \tag{18}$$

where $\forall\phi(\tau)\ h_{12}(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau)) < 0$.

Suppose that in formula (18) the variable t takes two values: t_k and t_{k+1}, $t_k < t_{k+1}$. Then we can define the numbers $\rho(t_k) = \rho_k > 0$, $\rho(t_{k+1}) = \rho_{k+1} > 0$, $x(t_k) = x_k > 0$, and $x(t_{k+1}) = x_k > 0$. Introduce the designation

$$\Delta(t) = \exp\left[\int_{t_0}^t (h_1(\cos\phi_1(\omega),\dots,\cos\phi_n(\omega)) + h_{12}(\cos\phi_1(\omega),\dots,\cos\phi_n(\omega))x(\omega))d\omega\right].$$

Then from formula (18) it follows that

$$\rho_{k+1} = \rho_0\Delta(t_{k+1}) + \Delta(t_{k+1}) \times \int_{t_0}^{t_{k+1}} \Delta(-\tau)x(\tau)h(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))d\tau,$$
$$\rho_k = \rho_0\Delta(t_k) + \Delta(t_k) \times \int_{t_0}^{t_k} \Delta(-\tau)x(\tau)h(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))d\tau,$$

and

$$\frac{\rho_{k+1}}{\rho_k} = \Delta(t_{k+1})\Delta(-t_k) + \frac{\Delta(t_{k+1})}{\rho_k} \times \int_{t_k}^{t_{k+1}} \Delta(-\tau)x(\tau)h(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))d\tau. \tag{19}$$

The first equation of system (4) on interval $[t_k, t_{k+1}]$ may be also written in the integral form as

$$x_{k+1} = x_k \exp(g(t_{k+1} - t_k)) + \int_{t_k}^{t_{k+1}} \exp(g(t_{k+1} - \tau)) \times [g_1(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))\rho(\tau) + g_{22}(\cos\phi_1(\tau),\dots,\cos\phi_n(\tau))\rho^2(\tau)]d\tau,$$
$$t_k < \tau < t_{k+1}. \tag{20}$$

Let $x(t_i) = x_i$, $\rho(t_i) = \rho_i$, $\phi(t_i) = \phi_i$, where t_i are roots of the first equation $\dot{x}(t_i) = 0$; i = 0,1,2,...[20,24].

We can also assume that the following variant takes place: t_k, t_{k+1}, t_{k+2},...are sequential maximum (or a point of inflection), minimum (or a point of inflection), and maximum (or a point of inflection) of the function $\rho(t)$.

Thus, from conditions $\dot{x}(t_i) = 0$ it follows that

$$x(t_k) = -\frac{1}{g}[g_1(\cos\phi_1(t_k), \ldots, \cos\phi_n(t_k))\rho(t_k) + g_{22}(\cos\phi_1(t_k), \ldots, \cos\phi_n(t_k))\rho^2(t_k)]. \quad (21)$$

Taking into account formula (21) and the known First Theorem About Mean Value we can rewrite formula (19) as

$$\rho_{k+1} \approx \Delta(t_{k+1})\Delta(-t_k)\rho_k + \frac{\Delta(t_{k+1})\Delta(-\xi_k)[g_1(\cos\phi_1(\xi_k), \ldots, \cos\phi_n(\xi_k))\rho(\xi_k) + g_{22}(\cos\phi_1(\xi_k), \ldots, \cos\phi_n(\xi_k))\rho^2(\xi_k)]}{-g} \times \int_{t_k}^{t_{k+1}} h(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))d\tau, \quad (22)$$

where $t_k < \xi_k < t_{k+1}$.

(d5) It is clear that in (22) we have $\Delta(t_{k+1})\Delta(-\xi_k) > 0$. In this case we can introduce the designations

$$\Delta(t_{k+1})\Delta(-t_k) = \theta > 0,$$
$$\frac{g_1(\cos\phi_1(t_k), \ldots, \cos\phi_n(t_k))\rho(t_k)}{-g} = \sigma,$$
$$\frac{g_{22}(\cos\phi_1(t_k), \ldots, \cos\phi_n(t_k))\rho(t_k)}{-g} = \psi > 0,$$
$$\int_{t_k}^{t_{k+1}} h(\cos\phi_1(\tau), \ldots, \cos\phi_n(\tau))d\tau = -\omega < 0,$$

where the functions θ, σ, ψ, and ω do not depend on k. (The last inequality is stipulated to those that in formula (22) we obtain $\lim_{t\to t_m^*} \rho(t_m^*) \to 0$ at some t_m^*, if and only if the second summand of this formula is negative. It is obvious that at the implementation of condition (iv) of Theorem 5 this restriction will be satisfied.)

Further, process (22) may be represented in the form

$$\rho_{k+1} = \theta \cdot \left((1+\sigma)\rho_k - \psi\omega\rho_k^2\right), k = 0, 1, \ldots \quad (23)$$

Let $1 + \sigma > 0$. Introduce the new variable $v_k = (\psi\omega/(1+\sigma)) \cdot \rho_k$. Then process (23) will be generated by known logistic map

$$v_{k+1} = \Psi(v_k, \varrho) \equiv \varrho \cdot v_k \cdot (1 - v_k), k = 0, 1, \ldots,$$

where $\varrho = \theta \cdot (1+\sigma) > 0$.

Let v_k^* be the minimal fixed point of 1D mapping $\Psi^{(k)}(v, \varrho) = \underbrace{\Psi(\Psi(\ldots, \Psi(v, \varrho), \varrho), \varrho)}_{k}$. It is known that if $\varrho = \varrho^* \in (3.84 \div 4)$, then the logistic map $\Psi(v, \varrho)$ is chaotic and $\lim_{k\to\infty} v_k^*(\varrho^*) = 0$. Hence, from here it follows that at some value ϱ^* of the parameter ϱ process (23) generates the subsequence $\rho_{m_1}, \ldots, \rho_{m_k}, \ldots$, for which $\rho_{m_k} = \lim_{t\to t_{m_k}^*} \rho(t_{m_k}^*) < \epsilon \approx 0$, $k \geq 1$. It means that in system (2) there is a chaotic dynamics.

Taking into account formula (17), we have $\Delta(t_{k+1})\Delta(-t_k) = \theta = \exp(E\rho_k^2 + F\rho_k + G)$. Then from here and formula (23) the discrete process (11) may be derived. (In our view formulas (10) and (11) justifies the name of the article.)

Consider the function $f(v) = \lambda v(1-v)\exp(-\mu v^2 + \nu v)$. Let $\dot{f}(v)$ be a derivative with respect to the variable v. We compute the maximum of this function on interval [0, 1]. For the solution of this task we will calculate roots of equation $\dot{f}(v) = 0$ on the interval [0, 1]. We have

$$\dot{f}(v) = 2\mu v^3 - (2\mu + \nu)v^2 + (\nu - 2)v + 1 = 0, \quad (24)$$

and the derivative $\dot{f}(0) > 0$, and the derivative $\dot{f}(1) < 0$. Thus, on interval [0, 1] there exists at least one positive root of equation (24). In obedience to the theorem of Descartes, equation (24) has two positive roots. From here and the condition $\dot{f}(1) < 0$ it follows that on interval [0, 1] there exists only one positive root v^*.

Let

$$\lambda^* = \frac{\exp(\mu v^{*2} - \nu v^*)}{v^*(1 - v^*)}.$$

Then $\forall\lambda \in [0, \lambda^*]$, we have $f(v)([0, 1]) \subset [0, 1]$. The state of chaos of the map $f(v)$ on the interval [0, 1] can be proved by the methods offered in [22]. Thus, the conclusions of all items (d1)–(d5) allow to complete the proof of Theorem 5 and its Corollary. □

Note that if numbers $v_0, v_1, \ldots, v_k, \ldots$ are small enough then formula (11) can be considered as the special case of formula (10). (Indeed, if $v > 0$ and $v \approx 0$, then $\exp(-v) \approx 1 - v$.)

Besides, from the conditions Theorem 5 it also follows that solutions $x(t)$ and $\rho(t)$ must be oscillating. It means that the linear part of system (2) have to have eigenvalues of opposite signs. In addition, for some values of parameters in this system there can be limit cycle. The presence of such limit cycle may be guaranteed by Theorem 3 under the conditions that one of eigenvalues of the matrix of the quadratic form $h_1(u_1, \ldots, u_n)$ is small negative and the magnitude $a_{10}^2 + \ldots + a_{n0}^2$ is also small enough. (Thus, for 3D systems the equilibrium point (0, 0, 0) is a saddle focus.)

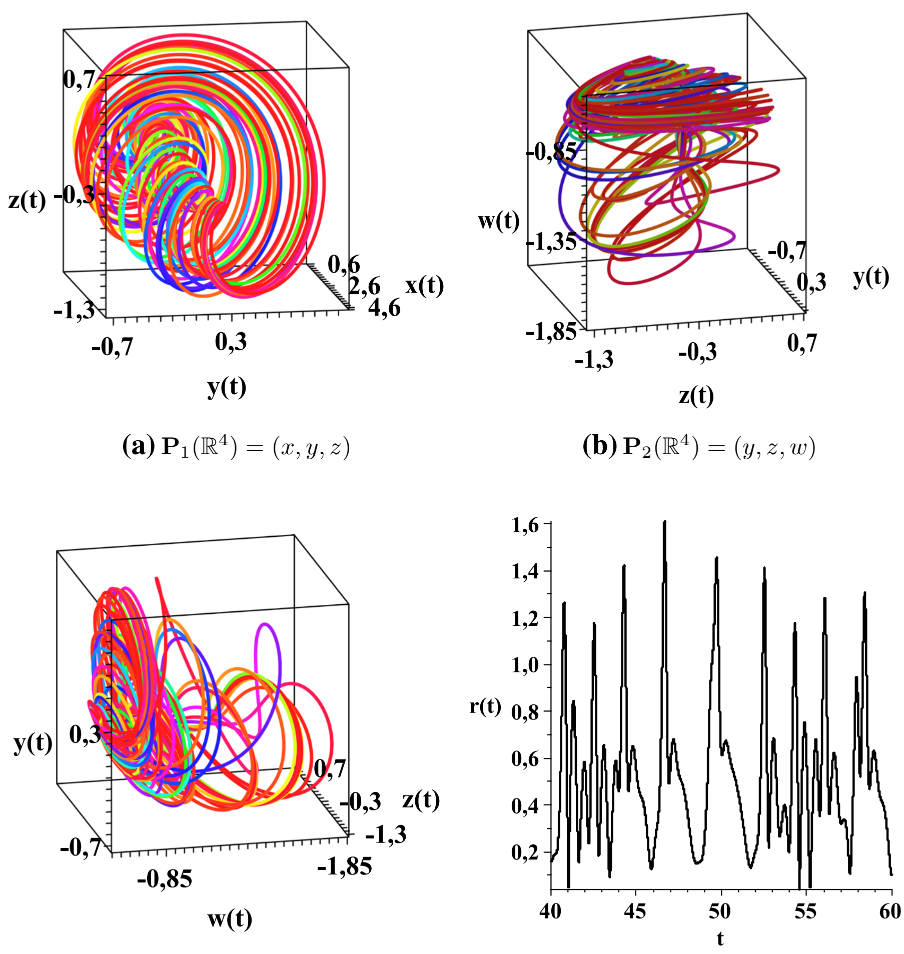
4 Example

Consider the following 4D generation of the known 3D system [23]:

$$\begin{cases} \dot{x}(t) = -2x(t) + 7y^2(t) + 13z^2(t) + 0.25w^2(t), \\ \dot{y}(t) = \mu x(t) + 7y(t) + 10z(t) - 3x(t)y(t), \\ \dot{z}(t) = -10y(t) + 7z(t) - 3x(t)z(t), \\ \dot{w}(t) = 0.55x(t) + 2z(t) + 2w(t) - 3x(t)w(t), \end{cases} \tag{25}$$

where $\mu = 1.81$.

For this system all conditions of Theorem 5 (and Theorem 4) are valid. On Fig. 1 the chaotic behavior of solutions of system (25) is shown.



(a) $\mathbf{P}_1(\mathbb{R}^4) = (x, y, z)$ **(b)** $\mathbf{P}_2(\mathbb{R}^4) = (y, z, w)$

(c) $\mathbf{P}_3(\mathbb{R}^4) = (z, w, x)$ **(d)**

Fig. 1 The projections (**a**)–(**c**) of the chaotic attractor of 4D system (25) on the different 3D subspaces in $\mathbb{R}^4 = (x, y, z, w)$ and dependence (**d**) of radius $r(t) = \rho(t) = \sqrt{y^2(t) + z^2(t) + w^2(t)}$ of this system on time t

As well as in [23] changing the parameter $\mu \in (0 \div 1.81)$ it is possible to observe the period double bifurcation (Feigenbaum's bifurcation). As a result of cascade of the period double bifurcations, a periodic solution of system (25) passes to chaotic.

5 Conclusion

The basic result of the present paper, which formulated in Theorem 5, supposes implementation of the condition $D(u_1, \ldots, u_n) < 0$. However, in works [24,25] the similar result, which supposes fulfillment for 3D quadratic systems of the more weak condition $D(u_1, u_2) = D(\cos\phi, \sin\phi) \leq 0$, was derived. (Moreover, in [24,25] it is shown that for Lorenz-like and Chen-like systems [19] the condition $D(\cos\phi, \sin\phi) \leq 0$ is hold.) Therefore, a question about generalizations of Theorem 5 on the case $D(u_1, \ldots, u_n) \leq 0$ remains opened.

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