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BRIEF COMMUNICATIONS

APPROXIMATION OF PERIODIC FUNCTIONS OF MANY VARIABLES BY FUNCTIONS OF A SMALLER NUMBER OF VARIABLES IN ORLICZ METRIC SPACES

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For periodic functions of many variables, we propose a method for their approximation in the Orlicz spaces $L_\varphi(\mathbb{T}^m)$. According to this method, the functions are approximated by the sums of functions of smaller number of variables each of which is piecewise-constant in one variable for fixed values of the other variables. A Jackson-type inequality is analyzed for these approximations in terms of the mixed module of continuity.

1. To pose the problem, we introduce the following notation:

$f(x),\ x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ is a real-valued function with period 1 in each variable;

$L_0 \equiv L_0(\mathbb{T}^m)$ is a set of measurable and almost everywhere finite functions on the main torus of periods $\mathbb{T}^m = [0,1]^m$;

Φ is a set of continuous and nondecreasing functions $\varphi\colon \mathbb{R}^1_+ \to \mathbb{R}^1_+,\ \varphi(0) = 0$;

$$L_\varphi \equiv L_\varphi(\mathbb{T}^m) = \left\{ f \in L_0; \|f\|_\varphi := \int\limits_{\mathbb{T}^m} \varphi(|f(x)|)dx < \infty \right\}$$ is a functional Orlicz class;

and

Ω is a set of functions ψ from Φ that are moduli of continuity, i.e., ψ are semiadditive functions.

Then $L_\psi \equiv L_\psi(\mathbb{T}^m)$ are metric spaces.

For $x \in \mathbb{R}^m$, let $(x; \hat{x}_j) := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m) \in \mathbb{R}^{m-1}$. By $\sum_m$ we denote a collection of functions $G(x),\ x \in \mathbb{T}^m$, of the form

$$G(x) = \sum_{j=1}^{m} g_j(x; \hat{x}_j), \quad g_j \in L_\varphi(\mathbb{T}^{m-1}).$$

Consider the problem of the best approximation of a function $f(x)$ of m variables by the sums of functions of $(m-1)$ variables in the space $L_\varphi(\mathbb{T}^m)$:

$$E(f)_\varphi = \inf\left\{ \big\|f(x) - G(x)\big\|_\varphi;\ G \in \Sigma_m \right\}. \tag{1}$$

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Generally speaking, the problem of finding the best functions G from Σ_m in (1) is quite complicated. In this connection, we study the approximation by functions that are possibly not the best functions from Σ_m but the method of their construction is simple. For these functions, we establish an upper bound of approximation in terms of the mixed module of continuity, which is exact for this method in the spaces L_ψ for all $\psi \in \Omega$ and in the spaces L_φ for a certain class of $\varphi \in \Phi$.

Let $\Delta^j_{t_j} f(x)$ be an increment of $f(x)$ in the variable x_j with step t_j. For $h = (h_1, \ldots, h_m) \in \mathbb{R}^m_+$, the quantity

$$\omega(f; h_1, \ldots, h_m)_\varphi = \sup\Big\{\big\|\Delta^1_{t_1} \ldots \Delta^m_{t_m} f\big\|_\varphi; |t_j| \le h_j,\ j = 1, \ldots, m\Big\}$$

is the mixed module of continuity of f.

For fixed $t \in \mathbb{T}^m$, we set

$$G_t(f; x) = f(x) - \Delta_{x_1 - t_1} \ldots \Delta_{x_m - t_m} f(t).$$

The function $G_t(f; x)$ is the sum of sections of $f(x)$ along the coordinate planes and belongs to Σ_m.

Theorem 1. *Suppose that φ belongs to Φ. Then, for all f from $L_\varphi(\mathbb{T}^m)$, there exists a value of the parameter $t_f \in \mathbb{T}^m$ such that*

$$\|f - G_{t_f}(f)\|_\varphi \le \omega\Big(f; \frac{1}{2}, \ldots, \frac{1}{2}\Big)_\varphi. \tag{2}$$

If φ satisfies the condition

$$\varphi(a) \ge \frac{\varphi(2^m)}{2^{2m}} a^2, \quad a = 1, 2, \ldots, 2^m, \tag{3}$$

then inequality (2) is strict in a sense that

$$\sup_{\substack{f \in L_\varphi(\mathbb{T}^m) \\ \omega\left(f; \frac{1}{2}, \ldots, \frac{1}{2}\right)_\varphi \ne 0}} \frac{\|f - G_{t_f}(f)\|_\varphi}{\omega\Big(f; \frac{1}{2}, \ldots, \frac{1}{2}\Big)_\varphi} = 1. \tag{4}$$

Proof. The existence of the parameter t_f follows from the estimation of averaging of the deviations of $G_t(f)$ over all t:

$$\int\limits_{t \in \mathbb{T}^m} \big\|f - G_t(f)\big\|_\varphi\, dt = \int\limits_{t \in \mathbb{T}^m} \big\|\Delta^1_{t_1} \ldots \Delta^m_{t_m} f\big\|_\varphi\, dt \le \omega\Big(f; \frac{1}{2}, \ldots, \frac{1}{2}\Big)_\varphi.$$

To construct extreme functions in (4), we use functions of one variable proposed by Yudin (see [1]).

Let $y \in \mathbb{R}^1$. For a prime number $q > 2$, we construct a partition of the period $[0, 1]$ by equidistant points $y_j = jq^{-1}$, $j = 0, 1, \ldots, q-1$, and define a 1-periodic function $f_q(y)$ by the conditions

$$f_q(y) = \left(\frac{j}{q}\right)$$

for $y \in (y_{j-1}, y_j]$ $j = 1, 2, \ldots, q-1$, and $f_q(y) = 0$ for $y \in (t_{q-1}, 1]$, where $\left(\dfrac{j}{q}\right)$ is the Legendre symbol [2, p. 70].

In the case where $m > 1$, for $x = (x_1, \ldots, x_m)$, we set

$$F_q(x) = \prod_{k=1}^{m} f_q(x_k).$$

Lemma 1. *Suppose that a function φ from Φ satisfies condition (3). Then*

$$\frac{\int_{t\in\mathbb{T}^m} \left\|\Delta^1_{t_1} \ldots \Delta^m_{t_m} F_q\right\|_{L_\varphi(\mathbb{T}^m)} dt}{\omega\left(F_q; \frac{1}{2}, \ldots, \frac{1}{2}\right)_{L_\varphi(\mathbb{T}^m)}} = 1 - c_q, \tag{5}$$

where $c_q > 0$ and $c_q \to 0$ as $q \to \infty$.

Proof. We first consider the case where $m = 1$.
Since [2, p. 82]

$$\sum_{r=0}^{q-1} \left(\frac{r}{q}\right)\left(\frac{r+j}{q}\right) = -1, \quad j = 1, \ldots, q-1,$$

for any y_j, we get

$$-\frac{1}{q} = \int_0^1 f_q(y) f_q(y + y_j) dy = \mu\big\{y \in [0,1]\colon\, f_q(y) f_q(y + y_j) = 1\big\}$$

$$- \mu\big\{y \in [0,1]\colon\, f(q) f_q(y + y_j) = -1\big\} + \frac{2}{q}$$

$$=\colon \mu^+ - \mu^- + \frac{2}{q}.$$

Since $\mu^+ + \mu^- + \dfrac{2}{q} = 1$, we conclude that $\mu^- = \dfrac{1}{2}\left(1 + \dfrac{1}{q}\right)$ and, therefore,

$$\left\|\Delta_{y_j} f_q\right\|_{L_\varphi(\mathbb{T}^1)} = \int_0^1 \varphi\big(|f_q(y + y_j) - f_q(y)|\big) dy$$

$$= \varphi(2) \cdot \frac{1}{2}\left(1 + \frac{1}{q}\right) + \varphi(1) \cdot \frac{2}{q}.$$

The function $\|\Delta_t f_q\|_{L_\varphi(\mathbb{T}^1)}$ is a piecewise monotone function of the argument t on the segments $[y_j, y_{j+1}]$. Hence,

$$\omega\left(f_q, \frac{1}{2}\right)_{L_\varphi(\mathbb{T}^1)} = \sup_j \|\Delta_{y_j} f_q\|_{L_\varphi(\mathbb{T}^1)} = \varphi(2) \cdot \frac{1}{2}\left(1 + \frac{1}{q}\right) + \varphi(1)\frac{2}{q}. \tag{6}$$

On the other hand, by using (3), we get

$$\|\Delta_t f_q\|_{L_\varphi(\mathbb{T}^1)} \geq \frac{\varphi(2)}{2^2} \int_0^1 \big|f_q(y) - f_q(y+t)\big|^2 dy$$

$$= \varphi(2)\frac{1}{2}\left(1 - \frac{1}{q} - \int_0^1 f_q(y) f_q(y+t)\,dy\right).$$

Further, in view of the fact that

$$\int_0^1 f_q(t)\,dt = 0,$$

we obtain

$$\int_0^1 \|\Delta_t f_q\|_{L_\varphi(\mathbb{T}^1)}\,dt \geq \varphi(2)\frac{1}{2}\left(1 - \frac{1}{q}\right). \tag{7}$$

Relation (5) with $m = 1$ follows from (6) and (7).

For $m > 1$, the proof is similar. For the sake of simplicity, we set $m = 2$. Then

$$\Delta^1_{t_1}\Delta^2_{t_2} F_q(x) = \Delta^1_{t_1} f_q(x_1)\Delta^2_{t_2} f_q(x_2),$$

$$\begin{aligned}
\big\|\Delta^1_{y_{j_1}}\Delta^2_{y_{j_2}} F_q\big\|_{L_\varphi(\mathbb{T}^2)} &= \int_0^1\int_0^1 \varphi\big(|\Delta_{y_{j_1}} f_q(x_1)|\,|\Delta_{y_{j_2}} f_q(x_2)|\big)\,dx_1 dx_2 \\
&= \varphi(2^2)\mu\big\{x \in [0,1]^2 : |\Delta_{y_{j_1}} f_q(x_1)|\,|\Delta_{y_{j_2}} f_q(x_2)| = 2^2\big\} \\
&\quad + \varphi(2)\mu\big\{x \in [0,1]^2 : |\Delta_{y_{j_1}} f_q(x_1)|\,|\Delta_{y_{j_2}} f_q(x_2)| = 2\big\} \\
&\quad + \varphi(1)\mu\big\{x \in [0,1]^2 : |\Delta_{y_{j_1}} f_q(x_1)|\,|\Delta_{y_{j_2}} f_q(x_2)| = 1\big\} \\
&= \varphi(2^2)\left(\frac{1}{2}\left(1 + \frac{1}{q}\right)\right)^2 + \varphi(2)\frac{2}{q}\left(1 + \frac{1}{q}\right) + \varphi(1)\left(\frac{2}{q}\right)^2 \\
&= \frac{1}{2^2}\varphi(2^2) + c_1(q),
\end{aligned} \tag{8}$$

where $c_1(q) \to 0$ as $q \to \infty$ and

$$\omega\left(F_q; \frac{1}{2}, \frac{1}{2}\right)_{L_\varphi(\mathbb{T}^2)} = \sup_{j_1, j_2} \big\|\Delta^1_{t_1}\Delta^2_{t_2} F_q\big\|_{L_\varphi(\mathbb{T}^2)} = \frac{1}{2^2}\varphi(2^2) + c_1(q),$$

$$\int_0^1\int_0^1 \left\|\Delta^1_{t_1}\Delta^2_{t_2}F_q)\right\|_{L_\varphi(\mathbb{T}^2)} dt_1\,dt_2$$

$$\geq \frac{1}{2^4}\varphi(2^2)\int_0^1\int_0^1\int_0^1\int_0^1 \left|\Delta_{t_1}f_q(x_1)\right|^2\Delta_{t_2}f_q(x_2)|^2\,dx_1\,dx_2\,dt_1\,dt_2$$

$$= \frac{1}{2^4}\varphi(2^2)\left(2\left(1-\frac{1}{q}\right)^2\right) = \frac{1}{2^2}\varphi(2^2)\left(1-\frac{1}{q}\right)^2. \tag{9}$$

Relation (5) follows from (8) and (9).

Lemma 1 and, hence, Theorem 1 are proved.

Note that condition (3) is satisfied for all $\varphi \in \Omega$ and also, e.g., for the functions $\varphi(x) = |x|^p$ with $p \in [1, 2]$.

2. We use Theorem 1 for the approximation of functions $f(x)$ from $L_\varphi(\mathbb{T}^m)$ by functions of the form

$$S(x) = \sum_{j=1}^m \varphi_j(x),$$

where each function $\varphi_j(x)$ is piecewise continuous in the variable x_j (for fixed values of the other variables).

For a given vector $\boldsymbol{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$, we construct uniform partitions of the coordinate axes: we split the jth coordinate of the torus of periods by the points

$$y_{r_j} = \frac{r_j}{n_j}, \quad r_j = 0, 1, \ldots, n_j,$$

into n_j segments of length $\dfrac{1}{n_j}$. Let $\chi_j(x_j)$ be the characteristic function of the segment $\left[0, \dfrac{1}{n_j}\right]$ and let the function $\varphi_j(x)$ have the form

$$\varphi_j(x) = \sum_{r_j=1}^{n_j} g_{r_j}(x; \hat{x}_j)\chi_j(x_j - y_{r_j}),$$

where $g_{r_j}(x; \hat{x}_j)$ are arbitrary functions from $L_\varphi(\mathbb{T}^{m-1})$ independent of the variable x_j.

Given $t \in \mathbb{T}^m$, let

$$f_t(x) := f(x + t)$$

be a shift of $f(x)$ by the parameter t. Consider the approximation

$$\int_{t\in\mathbb{T}^m} \|f_t - S_n\|_\psi\,dt$$

averaged over all shifts of f. As an approximating function S_n, we take

$$S_n(f_t; x) := f_t(x) - \sum_{r_1=1}^{n_1}\ldots\sum_{r_m=1}^{n_m} \Delta^1_{y_{r_1}-x_1}\ldots\Delta^m_{y_{r_m}-x_m} f_t(x)\chi_1(x_1 - y_{r_1})\ldots\chi_m(x_m - y_{r_m}).$$

It is easy to see that the function $S_n(f_t;x)$ belongs to the approximating subspace. Thus, for $m=2,$ we get

$$S_n(f_t;x) = -\sum_{r_2=1}^{n_2} f_t(x_1,y_{r_2})\chi(x_2-y_{r_2})$$

$$-\sum_{r_1=1}^{n_1} f_t(y_{r_1},x_2)\chi(x_1-y_{r_1}) + \sum_{r_1=1}^{n_1}\sum_{r_2=1}^{n_2} f_t(y_{r_1},y_{r_2}).$$

Theorem 2. *Assume that φ belongs to Φ. Then, for any $f\in L_\varphi(\mathbb{T}^m)$, the inequality*

$$\int\limits_{t\in\mathbb{T}^m} \big\|f_t - S_n(f_t)\big\|_\varphi\, dt \le \omega\Big(f;\frac{1}{n_1},\ldots,\frac{1}{n_m}\Big)_\varphi \tag{10}$$

is true. If φ satisfies condition (3), then inequality (10) is strict in the space $L_\varphi(\mathbb{T}^m)$:

$$\sup_{\substack{f\in L_\varphi(\mathbb{T}^m)\\ \omega\left(f;\frac{1}{n_1},\ldots,\frac{1}{n_m}\right)_\varphi \ne 0}} \frac{\int_{t\in\mathbb{T}^m} \|f_t - S_n(f_t)\|_\varphi\, dt}{\omega\Big(f;\frac{1}{n_1},\ldots,\frac{1}{n_m}\Big)_\varphi} = 1. \tag{11}$$

Proof. In each cube of the partition

$$\Pi_{r_1\ldots r_m} = \big\{x\in\mathbb{T}^m;\ x_i\in[y_{r_i},y_{r_{i+1}}),\ i=1,\ldots,m\big\},$$

the difference $f_t(x)-S_n(f_t;x)$ takes the form

$$f_t(x)-S_n(f_t;x) = \Delta^1_{y_{r_1}-x_1}\ldots\Delta_{y_{r_m}-x_m} f_t(x).$$

Therefore,

$$\int\limits_{t\in\mathbb{T}^m} \big\|f_t - S_n(f_t)\big\|_\varphi\, dt$$

$$= \sum_{r_1=1}^{n_1}\cdots\sum_{r_m=1}^{n_m} \int\limits_{t\in\mathbb{T}^m}\int\limits_{x\in\prod_{r_1\ldots r_m}} \varphi\big(|\Delta^1_{y_{r_1}-x_1}\ldots\Delta_{y_{r_m}-x_m} f(t+x)|\big)\, dx\, dt$$

$$= n_1\ldots n_m \int\limits_{h_1=0}^{\frac{1}{n_1}}\cdots\int\limits_{h_m=0}^{\frac{1}{n_m}} \big\|\Delta^1_{h_1}\ldots\Delta^m_{h_m} f\big\|_\varphi\, dh_1\ldots dh_m \le \omega\Big(f;\frac{1}{n_1},\ldots,\frac{1}{n_m}\Big)_\varphi .$$

To get the lower bound, we choose $f(x)=F_q(x)$ in (11) and apply Lemma 1.

Theorem 2 is proved.

The applied method of averaging over shifts proves to be efficient for the approximation of periodic functions of one variable by trigonometric polynomials in the spaces $L_p(\mathbb{T}^1),\ p \in (0,1)$ [3], and $L_\psi(\mathbb{T}^1)$ [4, 5]. In [6, 7], this method was used for the approximation by piecewise constant functions in $L_\psi(\mathbb{T}^1)$.

The obtained theorems can be regarded as one of possible multidimensional analogs of the results in [6]. Earlier, Lemma 1 was proved in the case $m = 1$ for the spaces $L_p(\mathbb{T}^1),\ p \in (0,2)$, in [1] and $L_\psi(\mathbb{T}^1),\ \psi \in \Omega$, in [6].

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