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RECURRENCE ANALYSIS OF TIME SERIES GENERATED BY 3D AUTONOMOUS QUADRATIC DYNAMICAL SYSTEM DEPENDING ON PARAMETERS

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For the wide class of 3D autonomous quadratic dynamical systems depending on parameters the sufficient conditions of boundedness of solutions of any system from this class are found. A connection between change of one of the parameters and a recurrence plot structure, which was built on the time series for any system of this class, is determined. Due to this connection it is possible to find bifurcation values of the parameter of any system from the considered class only on its time series without knowledge of differential equations of this system. Examples are given.

Keywords: 3D system of autonomous quadratic differential equations, recurrence plot, time series, period doubling bifurcation, limit cycle, chaotic attractor, delay-coordinate embedding technique, threshold.

1. Introduction

Let

$$x_0 = x(t_0), x_1 = x(t_1), ..., x_n = x(t_n) \tag{1.1}$$

be a finite sequence of numerical values of some scalar dynamical variable $x(t)$ measured with the constant time step Δt in the moments $t_i = t_0 + i\Delta t$; $x_i = x(t_i)$; $i = 0, 1, ..., n$. Sequence (1.1) is called a time series [1] – [3].

A common practice in chaotic time series analysis has been to reconstruct the phase space by utilizing the delay-coordinate embedding technique, and then to compute dynamical invariant quantities of interest such as unstable periodic orbits, the fractal dimension of the underlying chaotic set, and its Lyapunov spectrum. As a large body of literature exists on applying the technique to the time series from chaotic attractors [4] – [8], a relatively unexplored issue is its applicability to dynamical systems depending on parameters. Our focus will be concentrated on the analysis of influence of a period doubling bifurcation on behavior of the recurrence plot structure for the time series (1.1).

2. Embedding Method for Chaotic Time Series Analysis

Let sequence (1.1) be the time series. In principle, the measured time series comes from an underlying dynamical system that evolves the state variable in time according to a set of deterministic rules, which are generally represented by a set of differential equations, with or without the influence of noise. Mathematically, any such set of differential equations can be easily converted to a set of first-order, autonomous equations. The dynamical variables from all the first-order equations constitute the phase space, and the number of such variables is the dimension of the phase space, which we denote by $\mathbf{M}$. The phase space dimension can in general be quite large (in some cases it may be infinite) [5].

However, it often occurs that the asymptotic evolution of the system lives on a dynamical invariant set of a finite dimension. The assumption here is that the details of the system equations in the phase space and of the asymptotic invariant set that determines what can be observed through experimental probes, are unknown. The task is to estimate, based solely on one or few time series, practically useful statistical quantities characterizing the invariant set, such as its dimension, its dynamical skeleton, and its degree of sensitivity on initial conditions. The delay-coordinate embedding technique established by Takens [2], in particular, his famous embedding theorem guarantees that a topological equivalence of the phase space of the intrinsic unknown dynamical system can be reconstructed from the time series, based on which characteristics of the dynamical invariant set can be estimated.

Let

$$\dot{\mathbf{y}}(t) = \mathbf{F}(\mathbf{y}(t)), \mathbf{y} \in \mathbf{M} \subset \mathbb{R}^p \tag{2.1}$$

be the autonomous p-dimensional system of ordinary differential equations in the phase space $\mathbf{M}$.

We will consider that system (2.1) satisfies in the phase space $\mathbf{M}$ (an open region in $\mathbb{R}^p$) to the conditions of the known Cauchy Theorem about existence and uniqueness of solutions. Then for any initial condition $\mathbf{y}(0) = \mathbf{y}_0 \in \mathbf{M}$ it is possible uniquely to define the solution $\mathbf{y}(t)$ systems (2.1) on the formula $\mathbf{y}(t) = \mathbf{W}^t(\mathbf{y}_0)$, where $\mathbf{W}^t$ is an evolution operator. (A domain $\mathbf{G} \subset \mathbf{M}$ of the phase space $\mathbf{M}$ under action of the evolution operator passes, generally speaking, in another domain $\mathbf{G}_t = \mathbf{W}^t(\mathbf{G}) \subset \mathbf{M}$. If $\mathbf{G}_t = \mathbf{W}^t(\mathbf{G}) = \mathbf{G}$, then the domain $\mathbf{G}$ is called an invariant subset of the phase space $\mathbf{M}$ with respect to the action of the evolution operator $\mathbf{W}^t$.)

The compact invariant with respect to the evolution operator set $\mathbf{H} \subset \mathbf{M}$ is called attracting if there exists an open set $\mathbf{U} \subset \mathbf{M}$ containing $\mathbf{H}$ such that for almost all $\mathbf{y} \in \mathbf{U}$ $\lim_{t\to\infty} \mathbf{W}^t(\mathbf{y}) \in \mathbf{H}$. The indecomposable on two compact invariant subsets attracting set $\mathbf{H}$ is called an attractor.

It is known [6] that it is possible to get the attractor satisfactory image of a small dimension, if instead of the phase vector $\mathbf{y}(t)$ to use m-dimensional vectors derived from the time series (1.1) on the following formula:

$$\mathbf{x}_i = \begin{pmatrix} x_i \\ x_{i+1} \\ \vdots \\ x_{i+m-1} \end{pmatrix}; i \to 0, 1l, 2l, ..., il, ..., \tag{2.2}$$

where l is a positive integer.

Consider the m-dimensional autonomous dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{Q}(\mathbf{x}(t)), \mathbf{x} \subset \mathbb{R}^m, \tag{2.3}$$

for which the following conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}(t_0 + \tau) = \mathbf{x}_1, ...,\ \mathbf{x}(t_i) = \mathbf{x}(t_0 + i\tau) = \mathbf{x}_i$$

are fulfilled. (The magnitude $\mathbf{x}_i$ depends on $\mathbf{x}_0$ and τ, but it does not depend on t_0. We will especially emphasize that the map $\mathbf{Q} : \mathbb{R}^m \to \mathbb{R}^m$, determining the right part of systems (2.3), it is not known. In addition, it is clear that the role of the number l in (2.2) plays the number τ.) The magnitude τ is called a delay parameter of the time series (1.1).

Introduce the evolution operator $\mathbf{P}^t : \mathbb{R}^m \to \mathbb{R}^m$ of system (2.3). For any vector $\mathbf{x} \in \mathbb{R}^m$ the action of this operator in a coordinate form looks like:

$$\mathbf{P}^t(\mathbf{x}) = (P_0^t(\mathbf{x}), P_1^t(\mathbf{x}), ..., P_{m-1}^t(\mathbf{x}))^T.$$

Let $t = \tau$. Consider the sequence of real numbers

$$h_k = P_0^\tau(\mathbf{x}_k), h_{k+1} = P_1^\tau(\mathbf{x}_k), h_{k+2} = P_2^\tau(\mathbf{x}_k), \ldots, h_{k+m-1} = P_{m-1}^\tau(\mathbf{x}_k). \tag{2.4}$$

Introduce the new vector $\mathbf{z}_k$ under the formula: $\mathbf{z}_k = (h_k, h_{k+1}, ..., h_{k+m-1})^T$. Then there must be an operator $\mathbf{\Delta} : \mathbb{R}^m \to \mathbb{R}^m$ depending only on $\mathbf{Q}$ and τ such that $\mathbf{z} = \mathbf{\Delta}(\mathbf{x})$, where $\mathbf{x} = (x_i, x_{i+1}, ..., x_{i+m-1})^T$ is one of vectors (2.2).

Theorem 2.1. *[2] Let d be a dimension of the attractor $\mathbf{\Sigma}$ generated by system (2.3). Then for almost all $\tau > 0$ and $m \geq 2d+1$ the mapping $\mathbf{\Delta}$ will be continuous and one-to-one.*

Theorem 2.1 means that if in the space $\mathbb{R}^m$ to select the set $\mathbf{H}_k$ such that $\forall \mathbf{x}_k \in \mathbf{H}_k$ we have $\mathbf{\Delta}(\mathbf{x}_k) \in \mathbf{H}_k$, then on this set the map $\mathbf{\Delta}$ is invertible and $\forall k$ $\mathbf{x}_k = \mathbf{\Delta}^{-1}(\mathbf{z}_k)$.

By $\mathbb{N}$ denote the set of natural numbers.

Theorem 2.2. *[3] Let $i_1, i_2, ..., i_l, ...$ be an infinite sequence of positive integers. If system (2.3) is a dissipative then for any compact opened subset $\mathbf{\Phi} \subset \mathbb{R}^m$, any $\tau > 0$, and almost all $\mathbf{x} \in \mathbf{\Phi}$ the inclusion $(\mathbf{P}^\tau)^{i_l}(\mathbf{x}) = \underbrace{\mathbf{P}^\tau(\mathbf{P}^\tau(...(\mathbf{P}^\tau(\mathbf{x}))...))}_{i_l} \in \mathbf{\Phi}$; $l \in \mathbb{N}$ takes place.*

Thus, theorem 2.2 (which is sometimes called the Poincare recurrence theorem) asserts that in the phase space of the dissipative system any trajectory beginning from the almost liked point A of this space in some finite time (even very large) will pass as much as close to A.

Theorems 2.1 and 2.2 allowed to create the necessary research instrument which is used presently in the theory of the dynamic systems. Indeed, as the time series (1.1) has only a finite number of terms and, consequently, it is bounded, there is no justified arguments in order to assert that at the further measurements we will derive very large values of terms of this series. Further, the time series (1.1) describes the behavior of some phase variable of the explored dynamic process. If we assume that the number of such phase variables is finite, then it is possible to consider that there exists the evolution operator, which controls by the behavior of this dynamic system in some finite-dimensional space. In addition, most systems describing the dynamics of one or another processes in our world are dissipative. Thus, the use of Theorems 2.1 and 2.2 for description of dynamics of the dissipative finite-dimensional systems becomes more than justified.

Eckmann et al. [1] have introduced a tools which visualize the recurrence of states $\mathbf{x}_i$ in the phase space. Usually, the phase state does not have a dimension (it is more than two or three) which allows it to be pictured. Higher dimensional phase spaces can only be visualized by projection into the two or three dimensional subspaces.

Now by $\mathbf{x}(i)$ denote the point $\mathbf{x}_i = (x_i, x_{i+1}, ..., x_{i+m-1})^T$, which is built from the elements of the time series (1.1) describing the change of some scalar variable (or some coordinate of the vector variable, if a phase trajectory in m-dimensional space is considered); $i \to 0, 1l, 2l, ..., il$. (If $il + m - 1 > n$, then number i must be replaced by the number $k \to kl = il - [n/l]l$, where $[n/l]$ is an integer part of the number n/l. We will consider that $l = \tau$.)

Introduce in the first quadrant of the cartesian system of coordinates the graphic square matrix $T \in \mathbb{R}^{(n+1)\times(n+1)}$, which is built on the following algorithm: if point $\mathbf{x}(i)$ is close enough to the point $\mathbf{x}(j)$ (the concept of "closeness"will be defined below) then such points are called recurrence, and in the matrix T a black point with coordinates (i, j) are put. If point $\mathbf{x}(i)$ is not near to the point $\mathbf{x}(j)$, then in the matrix T no marks is done. The matrix F is called a recurrence plot of time series (1.1) [1] .

Let

$$\mathbf{R}_{ij} = \mathbf{\Theta}(\epsilon_i - \|\mathbf{x}_i - \mathbf{x}_j\|),\ \mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^m;\ i, j = 0, 1, ..., n,$$

be a real function accepting only two values: 0 and 1. (Here we have $\mathbf{\Theta}(\xi) = 1$, if $\xi \geq 0$ and $\mathbf{\Theta}(\xi) = 0$, if $\xi < 0$: it is the Heaviside function; $\|\mathbf{v}\| = \sqrt{v_1^2 + ... + v_m^2}$ is the Euclidian norm of the vector $\mathbf{v} \in \mathbb{R}^m$; ϵ_i is a radius of ball with a center in the point $\mathbf{x}_i$.)

In the future, it is possible to be restricted to the situation, when $\forall i, j\ \epsilon_i = \epsilon_j = \epsilon$. In this case positive number ϵ is called a recurrence threshold and we have symmetry of the recurrence plot with respect to the diagonal of the first quadrant.

Indeed, if point $\mathbf{x}_i$ is near to the point $\mathbf{x}_j$, then the reverse statement must be right: the point $\mathbf{x}_j$ is near to the point $\mathbf{x}_i$.

In the present paper we want to apply the instruments of the recurrence analysis for research of periodic trajectories in the dynamic systems described by 3D autonomous systems of differential equations. In order that such research was correct it is necessary to provide the boundedness of solutions of the explored systems.

3. Bounded Solutions of 3D Quadratic Dynamical Systems

Consider 3D real autonomous system

$$\dot{\mathbf{x}}(t) = H\mathbf{x} + \mathbf{f}(\mathbf{x}), \tag{3.1}$$

where $\mathbf{x} = (x, y, z)^T$; $H = \{h_{ij}\}$, $i, j = 1, ..., 3$, is a real (3×3)-matrix;

$$\mathbf{f}(\mathbf{x}) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z))^T \in \mathbb{R}^3$$

and

$$f_1(x, y, z) = a_{12}xy + a_{22}y^2 + a_{13}xz + a_{23}yz + a_{33}z^2,$$

$$f_2(x, y, z) = b_{12}xy + b_{22}y^2 + b_{13}xz + b_{23}yz + b_{33}z^2,$$

$$f_3(x, y, z) = c_{12}xy + c_{22}y^2 + c_{13}xz + c_{23}yz + c_{33}z^2$$

are real quadratic polynomials.

Suppose that the matrix

$$\begin{pmatrix} a_{12} & a_{13} \\ b_{12} & b_{13} \\ c_{12} & c_{13} \end{pmatrix}$$

has rank 1 or 2. Then by suitable linear transformations of variables $x \to x_1 + \alpha_1 y_1 + \alpha_2 z_1$ $(\alpha_1, \alpha_2 \in \mathbb{R})$, $y \to y_1$, and $z \to z_1$ system (3.1) can be represented in the same form (3.1), where $H \to \overline{H} = \{\overline{h_{ij}}\}$, $i, j = 1, ..., 3$, and

$$f_1(x_1, y_1, z_1) = \overline{a}_{22}y_1^2 + \overline{a}_{23}y_1z_1 + \overline{a}_{33}z_1^2,$$

$$f_2(x_1, y_1, z_1) = \overline{b}_{12}x_1y_1 + \overline{b}_{22}y_1^2 + \overline{b}_{13}x_1z_1 + \overline{b}_{23}y_1z_1 + \overline{b}_{33}z_1^2,$$

$$f_3(x_1, y_1, z_1) = \overline{c}_{12}x_1y_1 + \overline{c}_{22}y_1^2 + \overline{c}_{13}x_1z_1 + \overline{c}_{23}y_1z_1 + \overline{c}_{33}z_1^2,$$

and $\overline{b}_{12} \neq 0$ or $\overline{b}_{13} \neq 0$.

Thus, without loss of generality, we will study system (3.1) under the conditions

$$a_{12} = a_{13} = 0. \tag{3.2}$$

Introduce into system (3.1) (taking into account (3.2)) new variables ρ and ϕ under the formulas: $y = \rho \cos \phi$, $z = \rho \sin \phi$, where $\rho > 0$. Then, after replacement of variables and multiplication of the second and third equations of system (3.1) on the matrix

$$\begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -(\sin\phi(t))/\rho(t) & (\cos\phi(t))/\rho(t) \end{pmatrix},$$

we get

$$\begin{cases} \dot{x}(t) = s_{11}x(t) + (s_{12}\cos\phi(t) + s_{13}\sin\phi(t))\rho(t) + \\ \qquad (a_{22}\cos^2\phi(t) + a_{23}\cos\phi(t)\sin\phi(t) + a_{33}\sin^2\phi(t))\rho^2(t), \\ \dot{\rho}(t) = (s_{21}\cos\phi(t) + s_{31}\sin\phi(t))x(t) + [s_{22}\cos^2\phi(t) + s_{33}\sin^2\phi(t) + \\ \qquad (s_{32}+s_{23})\cos\phi(t)\sin\phi(t)]\rho(t) + \\ \qquad [b_{12}\cos^2\phi(t) + (b_{13}+c_{12})\cos\phi(t)\sin\phi(t) + c_{13}\sin^2\phi(t)]x(t)\rho(t) + \\ \qquad [b_{22}\cos^3\phi(t) + (b_{23}+c_{22})\cos^2\phi(t)\sin\phi(t) + \\ \qquad (b_{33}+c_{23})\cos\phi(t)\sin^2\phi(t) + c_{33}\sin^3\phi(t)]\rho^2(t), \\ \dot{\phi}(t) = (-s_{21}\sin\phi(t) + s_{31}\cos\phi(t))\dfrac{x(t)}{\rho(t)} + [s_{32}\cos^2\phi(t) - s_{23}\sin^2\phi(t) + \\ \qquad (s_{33}-s_{22})\cos\phi(t)\sin\phi(t)] - \\ \qquad [b_{13}\sin^2\phi(t) + (b_{12}-c_{13})\sin\phi(t)\cos\phi(t) - c_{12}\cos^2\phi(t)]x(t) - \\ \qquad [-c_{22}\cos^3\phi(t) + (b_{22}-c_{23})\cos^2\phi(t)\sin\phi(t) + \\ \qquad (b_{23}-c_{33})\cos\phi(t)\sin^2\phi(t) + b_{33}\sin^3\phi(t)]\rho(t). \end{cases} \tag{3.3}$$

Consider the first and second equations of system (3.3)

$$\begin{cases} \dot{x}(t) = s_{11}x + f(\cos\phi,\sin\phi)\rho + f_{22}(\cos\phi,\sin\phi)\rho^2, \\ \dot{\rho}(t) = g(\cos\phi,\sin\phi)x + h(\cos\phi,\sin\phi)\rho + g_{12}(\cos\phi,\sin\phi)x\rho + \\ \qquad g_{22}(\cos\phi,\sin\phi)\rho^2, \end{cases} \tag{3.4}$$

where ϕ is a real parameter;

$$\begin{aligned} f(\cos\phi,\sin\phi) &= s_{12}\cos\phi + s_{13}\sin\phi, \\ f_{22}(\cos\phi,\sin\phi) &= a_{22}\cos^2\phi + a_{23}\cos\phi\sin\phi + a_{33}\sin^2\phi, \\ g(\cos\phi,\sin\phi) &= s_{21}\cos\phi + s_{31}\sin\phi, \\ h(\cos\phi,\sin\phi) &= s_{22}\cos^2\phi + s_{33}\sin^2\phi + (s_{23}+s_{32})\cos\phi\sin\phi, \\ g_{12}(\cos\phi,\sin\phi) &= b_{12}\cos^2\phi + (b_{13}+c_{12})\cos\phi\sin\phi + c_{13}\sin^2\phi, \\ g_{22}(\cos\phi,\sin\phi) &= b_{22}\cos^3\phi + (b_{23}+c_{22})\cos^2\phi\sin\phi \\ &\quad + (b_{33}+c_{23})\cos\phi\sin^2\phi + c_{33}\sin^3\phi. \end{aligned}$$

Notice that the replacements of cartesian coordinates by polar is needed so that in system (3.4) both equations would be nonlinear with respect to the unknowns x and ρ. (The equation $\dot{\phi}(t) = ...$ in system (3.4) is not included.)

Let

$$\begin{aligned} &\Delta_1(\cos\phi,\sin\phi) \equiv s_{11}\cdot h(\cos\phi,\sin\phi) - f(\cos\phi,\sin\phi)\cdot g(\cos\phi,\sin\phi), \\ &\Delta_2(\cos\phi,\sin\phi) \equiv f_{22}(\cos\phi,\sin\phi)\cdot g_{12}(\cos\phi,\sin\phi), \\ &\Delta_3(\cos\phi,\sin\phi) \equiv g_{22}^2(\cos\phi,\sin\phi) + 4\Delta_2(\cos\phi,\sin\phi) \end{aligned}$$

be the bounded functions.

Theorem 3.1. *[9, 10] Let $s_{11} < 0$. Suppose also that $\forall \phi \in \mathbb{R}$ for system (3.4) the following conditions:*

(i) either $\Delta_1(\cos\phi, \sin\phi) \le 0$ or $\Delta_1(\cos\phi, \sin\phi)$ is a periodic alternating in sign on $(-\infty, \infty)$ function;

(ii) either $\Delta_2(\cos\phi, \sin\phi) \le 0$ and $\Delta_3(\cos\phi, \sin\phi) \le 0$ or $\Delta_2(\cos\phi, \sin\phi)$ and $\Delta_3(\cos\phi,\ \sin\phi)$ are periodic nonpositive functions,
are fulfilled.

Assume that the condition

$$\liminf_{t\to\infty} \rho(t) = 0 \tag{3.5}$$

is also valid. (From this condition it follows that $\forall \epsilon > 0$ there exists a numerical sequence $t_k \to \infty$ as $k \to \infty$ such that $\rho(t_k) < \epsilon$.)

Then in system (3.1) there is a chaotic dynamic.

Theorem 3.2. *Under the conditions of Theorem 3.1 the chaotic behavior of solutions of system (3.1) (or (3.3)) is generated by 1D iterated process*

$$\rho_{k+1} = \rho_k \exp\Big[\sigma - \chi\rho_k - \xi\rho_k^2\Big],\ \ \rho_k \ge 0;\ \ k = 0, 1, 2, ... \tag{3.6}$$

Here $\sigma > 0, \chi > 0, \xi > 0$.

Proof. Now we will consider that for some values of parameters system (3.1) has a periodic solution. (It means that system (3.3) also has the periodic solution.) Suppose also that $\phi_i(t_k) = \phi_i(t_0) + T \cdot k$, where $t_0 \ge 0$, $T \le N \cdot \pi$, and N is a positive integer; $k = 0, 1, 2, ...$; $i = 1, ..., n$.. Introduce the designations:

$$\begin{aligned}
s_{11} &= \xi_{11} < 0,\\
f(\cos\phi(t_k), \sin\phi(t_k)) &= \xi_{12} = const,\\
f_{22}(\cos\phi(t_k), \sin\phi(t_k) &= \zeta_{22} = const,\\
g(\cos\phi(t_k), \sin\phi(t_k)) &= \xi_{21} = const,\\
h(\cos\phi(t_k), \sin\phi(t_k)) &= \xi_{22} = const,\\
g_{12}(\cos\phi(t_k), \sin\phi(t_k)) &= \eta_{12} = const,\\
g_{22}(\cos\phi(t_k), \sin\phi(t_k)) &= \eta_{22} = const.
\end{aligned}$$

(Note that we have $\xi_{11}\xi_{22} - \xi_{12}\xi_{21} < 0$, $\eta_{12}\zeta_{22} \le 0$.)

Consider instead of system (3.4) the infinite sequence of systems of differential equations

$$\left\{\begin{array}{l}
\dot{x}_k(t) = \xi_{11}x_k + \xi_{12}\rho_k + \zeta_{22}\rho_k^2,\\
\dot{\rho}_k(t) = \xi_{21}x_k + \xi_{22}\rho_k + \eta_{12}x_k\rho_k + \eta_{22}\rho_k^2.
\end{array}\right. \tag{3.7}$$

(Here each of systems (3.7) is considering in the small neighborhood $\mathbb{O}_k$ at the point t_k: $t \in \mathbb{O}_k, k = 0, 1, 2, ...$ As initial conditions x_{k0}, ρ_{k0} for each of systems (3.7) the solutions of system (3.3) in the point t_k are appointed.)

Suppose that the time t_0 also satisfies the condition

$$\dot{x}(t_0) = \xi_{11}x_0 + \xi_{12}\rho_0 + \zeta_{22}\rho_0^2 = 0.$$

By virtue of periodicity of solutions of system (3.3), we can construct the sequence t_0, t_1,..., t_k,... such that for the first equation of system (3.7) the condition $\xi_{11}x_k + \xi_{12}\rho_k + \zeta_{22}\rho_k^2 = 0$ will be fulfilled $\forall t_k$, $k = 0, 1, 2, ...$. From here it follows that

$$x_k = -\frac{\xi_{12}\rho_k + \zeta_{22}\rho_k^2}{\xi_{11}};\ k = 0, 1, 2, ... \tag{3.8}$$

Consequently, $\forall t_k$ the second equation of system (3.7) will have the form

$$\dot{\rho}_k(t) = \frac{\xi_{11}\xi_{22} - \xi_{12}\xi_{21}}{\xi_{11}}\rho_k + \frac{\eta_{22}\xi_{11} - \eta_{12}\xi_{12} - \xi_{21}\zeta_{22}}{\xi_{11}}\rho_k^2 - \frac{\zeta_{22}\eta_{12}}{\xi_{11}}\rho_k^3$$

or

$$\dot{\rho}_k(t) = \rho_k(\beta - \gamma\rho_k - \delta\rho_k^2),$$

where $\beta = (\xi_{11}\xi_{22} - \xi_{12}\xi_{21})/\xi_{11} = \beta(t_k)$, $\gamma = (\eta_{22}\xi_{11} - \eta_{12}\xi_{12} - \xi_{21}\zeta_{22})/\xi_{11} = \gamma(t_k)$, and $\delta = \zeta_{22}\eta_{12}/\xi_{11} = \delta(t_k)$ are positive.

From here it follows that

$$\rho_k = c\exp\left(\int_0^{t_k}\Big[\beta(\tau) - \gamma(\tau)\rho(\tau) - \delta(\tau)\rho^2(\tau)\Big]d\tau\right)$$

and

$$\rho_{k+1} = c\exp\left(\int_0^{t_{k+1}}\Big[\beta(\tau) - \gamma(\tau)\rho(\tau) - \delta(\tau)\rho^2(\tau)\Big]d\tau\right).$$

Having excluded from two last equalities the constant c we get

$$\rho_{k+1} = \rho_k\exp\left(\int_{t_k}^{t_{k+1}}\Big[\beta(\tau) - \gamma(\tau)\rho(\tau) - \delta(\tau)\rho^2(\tau)\Big]d\tau\right).$$

Let the bounded positive function $\theta(t)$ be a monotone decreasing on interval $[t_i, t_{i+1}]$, and let it be a monotone increasing on interval $[t_{i+1}, t_{i+2}]$. Then we have (Second Theorem About Mean Value):

$$\int_{t_i}^{t_{i+2}} h(\phi(\tau))\cdot\theta(\tau)d\tau = \int_{t_i}^{t_{i+1}} h(\phi(\tau))\cdot\theta(\tau)d\tau + \int_{t_{i+1}}^{t_{i+2}} h(\phi(\tau))\cdot\theta(\tau)d\tau =$$

$$= \theta(t_i + 0)\int_{t_i}^{\xi} h(\phi(\tau))d\tau + \theta(t_{i+2} - 0)\int_{\zeta}^{t_{i+2}} h(\phi(\tau))d\tau, \tag{3.9}$$

where $t_i \le \xi \le t_{i+1}$, $t_{i+1} \le \zeta \le t_{i+2}$. Hence, from (3.9) it follows that

$$\int_{t_i}^{t_{i+2}} h(\phi(\tau))\cdot\theta(\tau)d\tau = p_i\theta_i + p_{i+2}\theta_{i+2}, \tag{3.10}$$

where magnitudes $p_i = \int_{t_i}^{\xi} h(\phi(\tau))d\tau, p_{i+2} = \int_{\zeta}^{t_{i+2}} h(\phi(\tau))d\tau$ can have any signs.

Now let the function $\rho(t)$ be periodic. Then in (3.10) we have $\theta_i = \theta_{i+2}$. From here it follows that

$$\rho_{i+1} = \rho_i \exp\Big[\sigma_i - \chi_i\rho_i - \xi_i\rho_i^2\Big],$$

$$\rho_{i+2} = \rho_{i+1} \exp\Big[\sigma_{i+1} - \chi_{i+1}\rho_{i+2} - \xi_{i+1}\rho_{i+2}^2\Big],$$

and, therefore, we have

$$\rho_{i+2} = \rho_i \exp\Big[\sigma_i + \sigma_i - (\chi_i + \chi_{i+1})\rho_i - (\xi_i + \xi_{i+1})\rho_i^2\Big];\ i = 0, 2, 4, ... \tag{3.11}$$

It is clear that the quadratic function $\theta(\rho) = \beta - \gamma\rho - \delta\rho^2$ is decreasing on the interval $[0, \infty)$. In addition, $\forall i$ the magnitudes $\sigma_i+\sigma_{i+1}$, $\chi_i+\chi_{i+1}$, and $\xi_i+\xi_{i+1}$ do not depend on i (it is constant). Let $\sigma = \sigma_i+\sigma_{i+1}$, $\chi = \chi_i+\chi_{i+1}$, and $\xi = \xi_i+\xi_{i+1}$. Then taking into account formula (3.11) we can derive the following formula

$$\rho_{k+1} = \rho_k \exp\Big[\sigma - \chi\rho_k - \xi\rho_k^2\Big], k = i/2 = 0, 1, 2, ...,$$

where

$$\sigma(\tau) = \int_{t_k}^{t_{k+1}} \beta(\tau)\cdot d\tau > 0,\ \xi = \int_{t_k}^{t_{k+1}} \delta(\tau)\cdot d\tau > 0,\ \chi = \int_{t_k}^{t_{k+1}} \gamma\cdot d\tau > 0.$$

The proof is finished. □

A. N. Sharkovsky (see [11]) introduced the following new ordering of all positive integers:

$$\begin{gathered}
3 \succcurlyeq 5 \succcurlyeq 7 \succcurlyeq 9 \succcurlyeq 11 \succcurlyeq 13 \succcurlyeq \dots \\
\succcurlyeq 2\cdot 3 \succcurlyeq 2\cdot 5 \succcurlyeq 2\cdot 7 \succcurlyeq 2\cdot 9 \dots \\
\succcurlyeq 2^2\cdot 3 \succcurlyeq 2^2\cdot 5 \succcurlyeq 2^2\cdot 7 \succcurlyeq 2^2\cdot 9 \succcurlyeq \dots \\
. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
\succcurlyeq 2^n\cdot 3 \succcurlyeq 2^n\cdot 5 \succcurlyeq 2^n\cdot 7 \succcurlyeq \dots \\
\succcurlyeq 2^{n+1}\cdot 3 \succcurlyeq 2^{n+1}\cdot 5 \succcurlyeq 2^{n+1}\cdot 7 \succcurlyeq \dots \\
\succcurlyeq 2^{n+1} \succcurlyeq 2^n \succcurlyeq 2^{n-1} \succcurlyeq \dots \succcurlyeq 2^2 \succcurlyeq 2 \succcurlyeq 1.
\end{gathered}$$

The first row the Sharkovsky ordering is all prime numbers except 1. The last row is called Feigenbaum's scenario of the period doubling bifurcation.

Let $f(x) : [0, \infty) \to [0, \infty)$ be a real continuous function. The number $x \in [0, \infty)$ is called a cycle of period k if $\underbrace{f(f(...f(x)...))}_{k} = x$.

Theorem 3.3. *[11] If the continuous function $f(x) : [0, \infty) \to [0, \infty)$ has a cycle of period n with $n \succcurlyeq k$ (in the sense Sharkovsky's ordering indicated above), then it also has a cycle of period k. If the map $f(x)$ has 3-cycle then it map is chaotic.*

Theorems 3.1 – 3.3 have the following value for the study of time series.

1. From Theorem 3.1 it follows that if a time series is got from the solutions of system (3.1), then his terms will not accept as much as desired large values.

2. Theorem 3.2 shows the structure of discrete maps which generate the chaotic behavior of the time series.

3. Assume that the real 1D map f depends on some parameters and looks like $f(x, \sigma, \chi, \xi) = x \cdot \exp(\sigma - \chi x - \xi x^2)$ (see (3.6)). Then Theorem 3.3 shows a transition scenario from the regular (periodic or quasi-periodic) to the chaotic behavior of the time series as a result of change of the parameters of system (3.1).

4. Examples

In the examples of considered below we will assume that all conditions of Theorem 3.1 are fulfilled. Then the chaotic behavior of solutions of system (3.1) (or (3.3)) is generated by 1D iterated process (3.6).

1. Feigenbaum's scenario is a demonstrating of the period doubling bifurcation as a result of change of the real parameter μ.

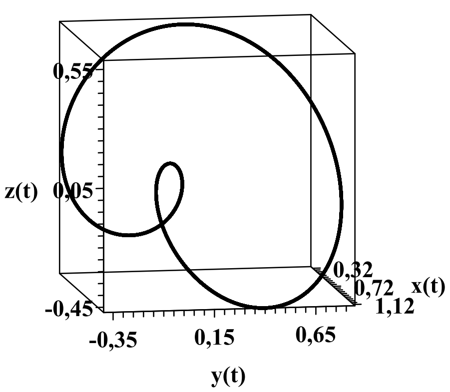
Consider the following system [10]:

$$\begin{cases} \dot{x}(t) = -2x(t) + 7y^2(t) + 4z^2(t), \\ \dot{y}(t) = \mu x(t) + 4y(t) + 7z(t) - 6x(t)y(t), \\ \dot{z}(t) = \mu x(t) - 7y(t) + 4z(t) - 6x(t)z(t), \end{cases} \tag{4.1}$$

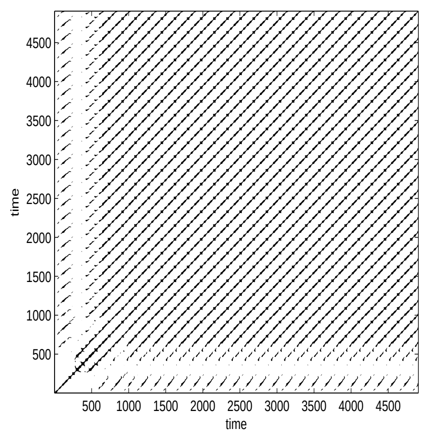
In the polar coordinates system (4.1) takes the form (see (3.3))

$$\begin{cases} \dot{x}(t) = -2x + (7\sin^2\phi + 4\cos^2\phi) \cdot \rho^2, \\ \dot{\rho}(t) = \mu \cdot (\cos\phi + \sin\phi) \cdot x + 4\rho - 6x\rho, \\ \dot{\phi}(t) = -7 - \mu \cdot (\cos\phi - \sin\phi) \cdot x/\rho. \end{cases} \tag{4.2}$$

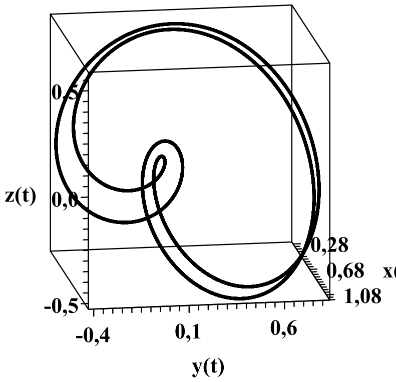
Below four steps Feigenbaum's scenario and transition to chaos are shown.



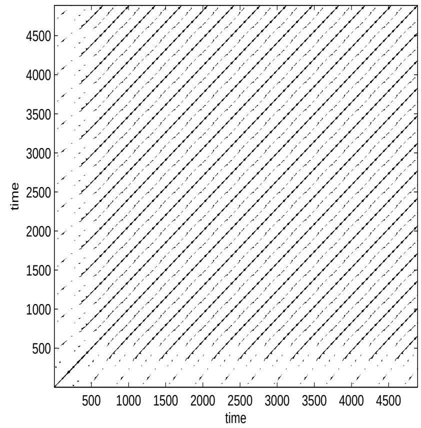
(a1) $\mu = 1.20$. This is 1-cycle



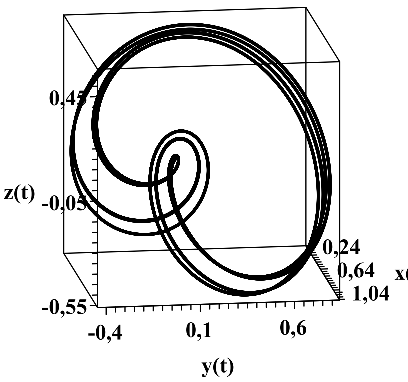
(b1)



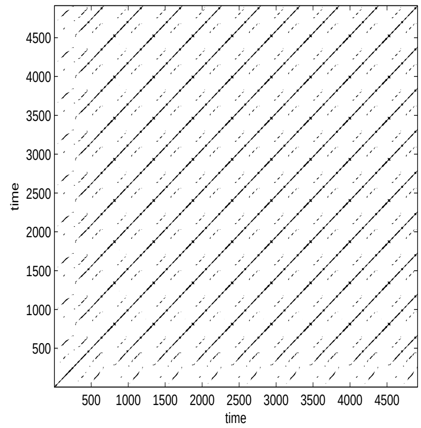
(a2)$\mu = 1.30$. This is 2-cycle



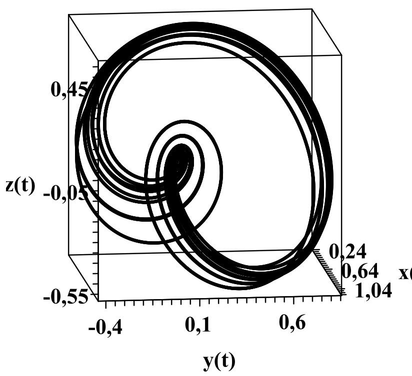
(b2)



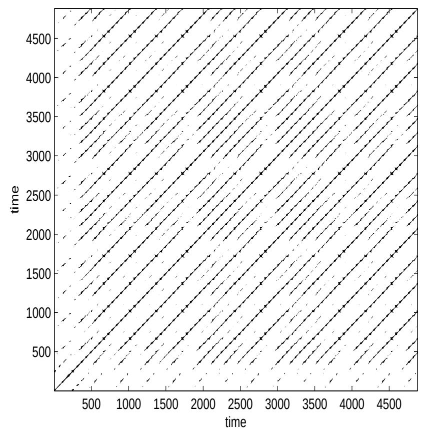
(a3)$\mu = 1.35$. This is 4-cycle



(b3)

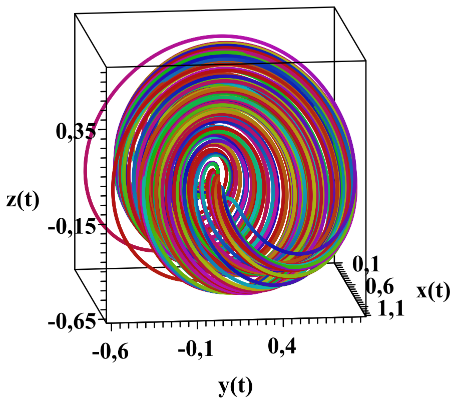


(a4)$\mu = 1.38$. This is 8-cycle

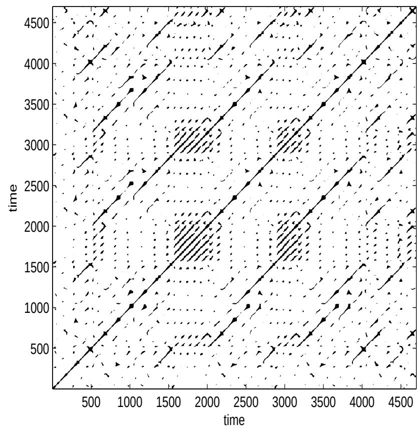


(b4)

(thus, the scenario of period doubling bifurcation is realized: 16-cycle, 32-cycle,...)



(a5)$\mu = 1.57$. This is a chaos



(b5)

Fig. 1. The phase portraits (a1) – (a5) and the recurrence plots (b1) – (b5) of system (4.1) are at different values of μ. There is a period doubling bifurcation.

It is visible from Figure 1 that at different values μ a structure of the recurrence plots remains identical. Frequency of appearance of diagonal lines changes only.

2. The Sharkovsky ordering.

Consider the following system [12]:

$$\begin{cases} \dot{x}(t) = ax(t) + y(t) \cdot (a_{22}y(t) + a_{23}z(t)), \\ \dot{y}(t) = h_{21}x(t) + by(t) + cz(t) - d \cdot x(t) \cdot (a_{22}y(t) + a_{23}z(t)), \\ \dot{z}(t) = h_{31}x(t) - cy(t) + bz(t), \end{cases} \tag{4.3}$$

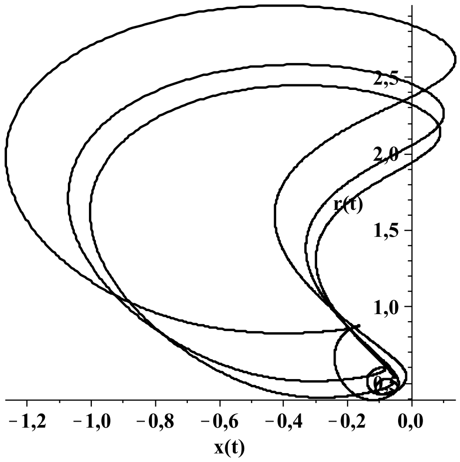
In the polar coordinates system (4.3) takes the form

$$\begin{cases} \dot{x}(t) = ax + \cos\phi \cdot (a_{22}\cos\phi + a_{23}\sin\phi) \cdot \rho^2, \\ \dot{\rho}(t) = (h_{21}\cos\phi + h_{31}\sin\phi) \cdot x + b\rho - d\cos\phi \cdot (a_{22}\cos\phi + a_{23}\sin\phi) \cdot x\rho, \\ \dot{\phi}(t) = -c + (h_{31}\cos\phi - h_{21}\sin\phi) \cdot x/\rho + d\sin\phi \cdot (a_{22}\cos\phi + a_{23}\sin\phi) \cdot x. \end{cases} \tag{4.4}$$

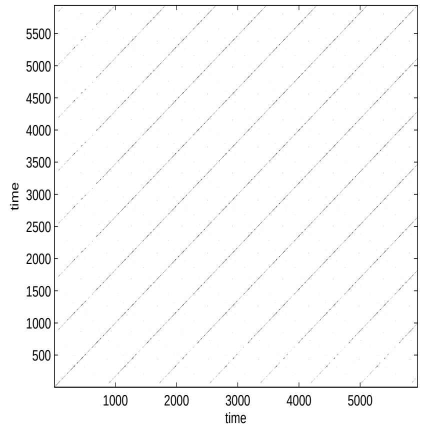
We note that on terminology of paper [12] system (4.3) is essentially singular. It means that condition (3.5) is difficult to attain. Therefore in order that this condition will be fulfilled it was required to explore the system for which $h_{21} = h_{31} = 0$. In this case system (4.3) has periodic solutions. Consequently changing the parameters h_{21} and h_{31} it is possible to get chaotic systems (4.3). It is especially important at such changes to get cycles of multiple 3 (3, 6, 12...) in system (4.3) (or (4.4)). In obedience to Theorem (3.3) it means beginning of transition of the system state from periodic to chaotic.

Below three steps Sharkovsky's ordering and transition to chaos are shown.

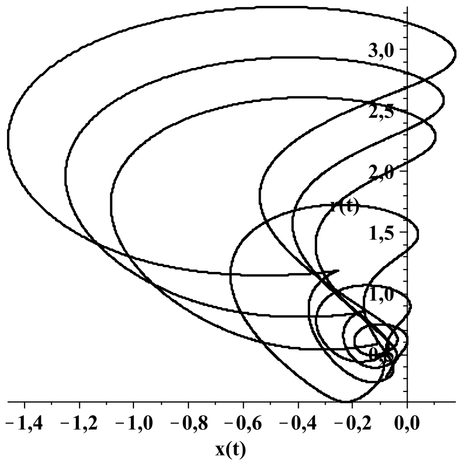
Assume that $a = -2.2$, $b = 0.5$, $c = 1.5$, $d = 1$, $a_{22} = -1, a_{23} = 1$, $h_{21} = -3, h_{31} = \mu$.



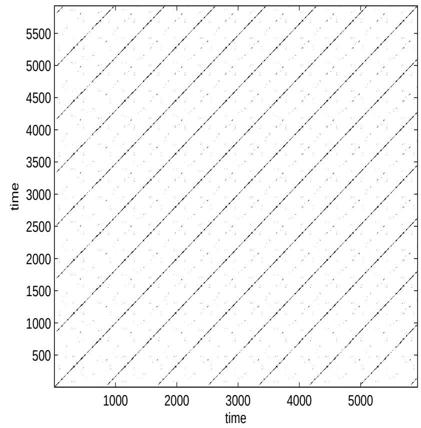
(a1)$\mu = 1.634$. This is 3-cycle



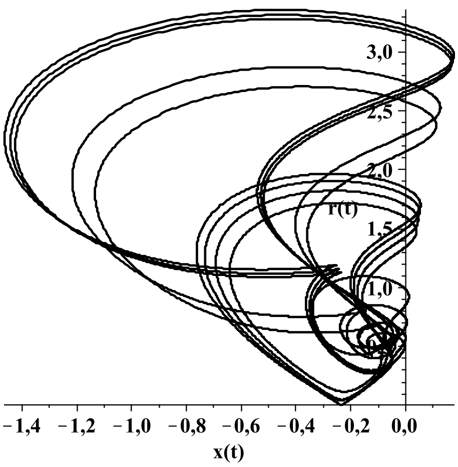
(b1)



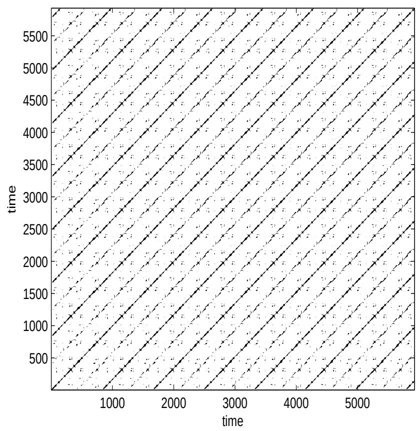
(a2)$\mu = 1.642$. This is 6-cycle



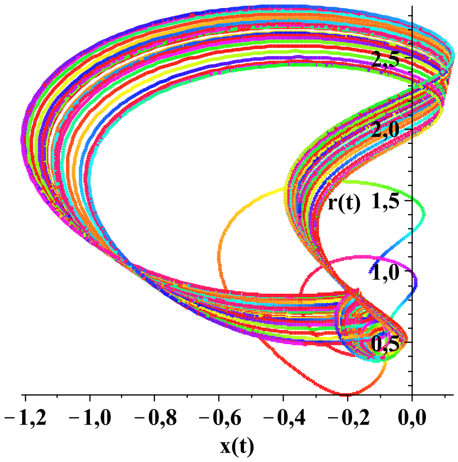
(b2)



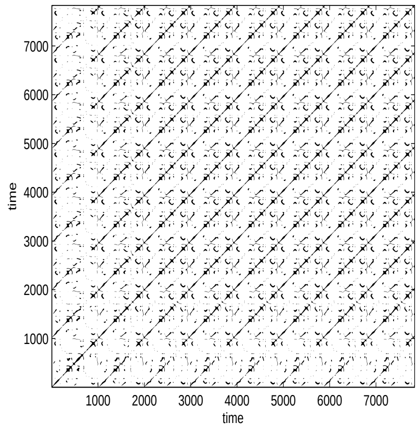
(a3)$\mu = 1.6422$. This is 12-cycle



(b3)



(a4)$\mu = 1.7202$. This is a chaos



(b4)

Fig. 2. The phase portraits (a1) – (a4) and the recurrence plots (b1) – (b4) of system (4.4) are at different values of μ. There is the Sharkovsky ordering.

5. Conclusion and analysis of results

At the analysis of recurrence plots a big role plays the length of diagonal lines (we will emphasize that on recurrence plots the length of line characterizes a response time of trajectory in some region of phase space). This description shows that the time interval in during of which some part of phase trajectory passes parallel to other part of the same trajectory. In other words, the trajectory repeats itself passing through the same region of phase space in different intervals of time. Thus, if some diagonal lines parallels to the main diagonal, it means periodicity of this process in some region of phase space.

As it was said above Theorems (3.1) – (3.3) play a large role in research of the chaotic systems behavior. The role of Theorem (3.1) in this research is clear. It remained to determine the role of Theorems (3.2) and (3.3).

First we note that these theorems explain the reason of appearance of chaos in the smooth dynamic system. This reason is connected with the generation in system (4.4) of the iterative process (3.6).

Further, in order to prove the presence of chaos in system (4.4) it is necessary to confirm existence of 3-cycle at some values of the parameter μ. On Figure 2 it is visible that there exists moments t_k such that for these moments $\rho(t_k) \approx 0$.

In obedience to Theorem (3.1) it means appearance of chaos in system (4.4). Theorem (3.3) asserts that the reason of origin of chaos is existence of 3-cycles in system (4.4). It is well visible on Figure 2. In addition, on the same figure the Sharkovsky ordering (3, 6, 12,...) is shown.

It is also necessary to say that the recurrence analysis well confirmed doubling of period on Figures 1 and 2. Indeed, both in Feigenbaum's scenario and in

Sharkovsky's scenario with growth of the parameter μ the distance between two parallel lines of one texture located by near is strictly doubled from some base size.

Thus, we can assert that the recurrence analysis is an effective tool for the study of time series of a different character.

References

1. *J.-P. Eckmann, S. Kamphorst, D. Ruelle*, *Recurrence plots of dynamical systems*, Europhysics Letters, **4**(9)(1987), 973 – 977.
2. *F. Takens*, *Detecting strange attractors in turbulence*, Lecture Notes in Mathematics, Springer, Berlin, **898**(1981), 366 – 381.
3. *H. Poincare*, *Sur la probleme des trois corps et les equations de la dynamique*, Acta Mathematica, **13**(1890), 1 – 270.
4. *J. P. Zbilut, C. I. Webber*, *Embeddings and delays as derived from quantification of recurrence plots*, Physics Letters, **A171**(1992), 199 – 203.
5. *Y.-C. Lai, N. Ye*, *Recent developments in chaotic time series analysis*, International Journal of Bifurcation and Chaos, **13**(6)(2003), 1383 – 1422.
6. *N. Marwan, M. Romano, M. Thiel, J. Kurths*, *Recurrence plots for the analysis of complex systems*, Physics Reports, **438**(2007), 237 – 329.
7. *N. Marwan*, *How to avoid pitfalls in recurrence plot based data analysis*, International Journal of Bifurcation and Chaos, **21** (4)(2011), 1003 – 1017.
8. *A. P. Shultz, Y. Zou, N. Marwan, M. T. Turvey*, *Local minima-based recurrence plots for continuous dynamical systems*, International Journal of Bifurcation and Chaos, **21** (4)(2011), 1065 – 1075.
9. *V. Ye. Belozyorov*, *Exponential-algebraic maps and chaos in 3D autonomous quadratic systems*, International Journal Bifurcation and Chaos, **25**(4)(2015), ID 1550048, 1–24.
10. *V. Ye. Belozyorov*, *Research of chaotic dynamics of 3D autonomous quadratic systems by their reduction to special 2D quadratic systems*, Mathematical Problems in Engineering, **2015**(2015), ID 271637, 1–15 pages.
11. *J. Andres, P. Snyrychova, P. Szuca*, *Sharkovskiy's Theorem for connectivity G_δ-relations*, International Journal of Bifurcation and Chaos, **16**(8)(2006), 2377 – 2393.
12. *V. Ye. Belozyorov*, *Invariant approach to existence problem of chaos in 3D autonomous quadratic dynamical systems*, International Journal of Bifurcation and Chaos, **26**(1)(2016), Article ID 1650012, 1–14.

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