hodograpf; transit curves; spiral line; guidepaths

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MATHEMATICAL MODEL OF A PATH AND HODOGRAPF OF SURFACE TRANSPORT

Summary. With the increase in speed of wheeled vehicles, such lead nations follow their research on determining new forms of transit curves. Transit curves proposed from the viewpoint of heuristic or criteria ideas should meet true guide path of transport vehicle at both constant and alternative speed.

It is proposed to form a plan, longitudinal and cross-sectional profile of highways in junctions and turns in the form of ruled surface (Shukhov's surface) which components are in a normal plane of true trihedr on of motion path and guide path selected as a part of spiral line covering nine variable parameters determined on required boundary conditions imposed on the curve value too if motion modes are specified.

It is set as specified both location and speed of a vehicle during input/output moments within considered curved highway area. It is required to determine hodograph corresponding specified mode of the vehicle movement along the highway considered as a motion trajectory.

Hodograph of vehicle movement depending upon various modes – accelerated, decelerated, with constant speed, injunctions and turns, within straight and curved areas, while sloping and rising can be synthesized in a class of proposed spiral lines (guidepaths) corresponding to true, undisturbed trajectories of a vehicle.

1.INTRODUCTION

With the increase in speed of wheeled vehicles, such lead nations as Germany, France, Japan and others follow their research on determining new forms of transit curves in the shape ofcubic parabola, sinusoid lemniscates, three- or four-leafed rose as well as pseudospiralbeing in some cases circumference, logarithmic spiral, clothoid etc. [3,4]. Transit curves proposed from the viewpoint of heuristic or criteria ideas should meet true guide path of transport vehicle at both constant and alternative speed [1, 3].

It is proposed to form a plan, longitudinal and cross-sectional profile of highways in junctions and turns in the form of ruled surface (Shukhov's surface) which components are in a normal plane of true trihedron of motion path in perpendicular to a projection of resulting vector of inertia forces including gravitation to this true plane in each point of guide path selected as a part of spiral line covering nine variable parameters determined on required boundary conditions imposed on the curve value too if motion modes are specified [5 - 7].

2. THE PROBLEM FORMULATION

It is set as specified both location and speed of a vehicle during input/output moments within considered curved highway area. It is required to determine hodograph corresponding specified mode of the vehicle movement along the highway considered as a motion trajectory.

3. GUIDEPATH OF ROAD SURFACE

Mathematical model of road surface of a highway in junctions and turns is built in the form of ruled surface (Shukhov's surface). Guidepath (transition curve) of this surface is selected as corresponding to true trajectory of a vehicle motion considered in this term as a material point. This trajectory is shown in parametric form where time of the vehicle turn motion (hodograph) is taken as a parameter.

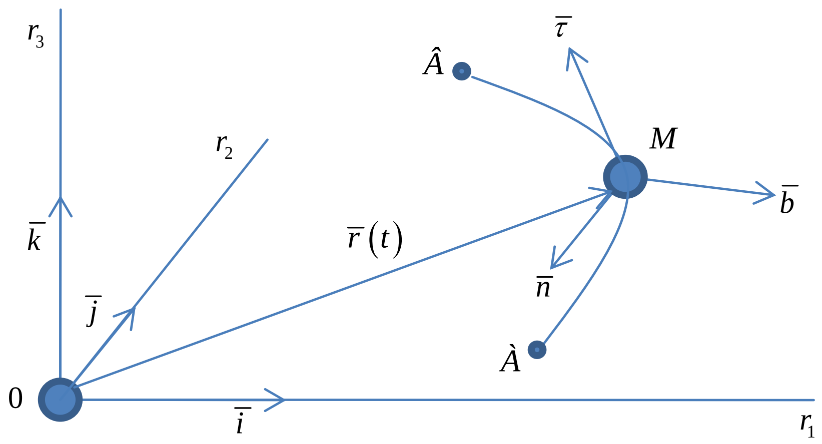
Hodograph corresponding to true motion trajectory should be found in the class of spiral lines specified in fixed (earth) coordinates as follows:

$$\bar{r}(t) = \|\rho_0 \rho_1 \rho_2 \rho_3\| \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \left(\bar{i}\cos\omega t + \bar{j}\sin\omega t\right) + \bar{k}\,\|h_0 h_1 h_2 h_3\| \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix}$$

where $\rho_i\, h_i\,(i = 0,1,2,3)$ are varied parameters determined according to specified boundary conditions;

ω is average angular turn speed equal to $\omega = \dfrac{\varphi_0}{t_0}$. Here φ_0 is complete turn angle; and t_0 is the time required for the turn.

The figure shows the vehicle pass as the trajectory of vehicle motion within curved area; it corresponds to proposed hodograph:



Where $\bar{i}, \bar{j}, \bar{k}$ are unitary vectors of earth (inertial) coordinate system;

$\bar{\tau}, \bar{n}, \bar{b}$ are unitary vectors of true trajectory trihedron.

Represent hodograph in a well-known general form:

$$\bar{r}(t) = \bar{i}\, r_1 + \bar{j}\, r_2 + \bar{k}\, r_3\,,$$

Where the vector components are assumed to be set as follows:

$$r_1 = \left\| \rho_0 \rho_1 \rho_2 \rho_3 \right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| \cos \omega t \,, \quad r_2 = \left\| \rho_0 \rho_1 \rho_2 \rho_3 \right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| \sin \omega t \,, \quad r_3 = \left\| \rho_0 \rho_1 \rho_2 \rho_3 \right\| \left\| \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \right\| .$$

4. VEHICLE HODOGRAPH WITHIN STRAIGHT AND CURVED HIGHWAY AREAS DEPENDING UPON VARIOUS MOTION MODES

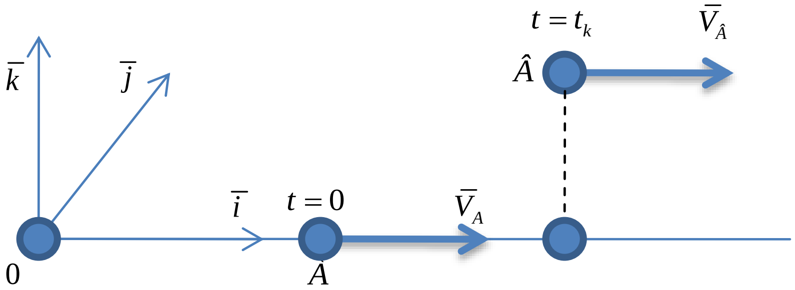
Earth coordinate system considers a vehicle hodograph as material points within highway areas with/without slope, without turn and in it, at motion modes with constant speed, deceleration, and acceleration.

4.1. Construction of hodograph of a vehicle within a highway area having rise (slope), without turn

Boundary conditions are:

$$\bar r_A = \bar i \, r_{1A} , \qquad \bar r_B = \bar i \, r_{1B} + \bar k \, r_{3B} ,$$
$$\bar V_A = \bar i \, V_{1A} , \qquad \bar V_B = \bar i \, V_{1B} + \bar k \, V_{3B}$$

Following Figure demonstrate them:



Rise (slope) characterizes: $r_{3B} \neq 0$, non availability of turn means: $\omega = 0$.
Congruence of speeds at the beginning/end of highway area is reduced to the conditions:

$$V_{1A} = V_{1B} , \qquad V_{3B} = 0 , \qquad V_{3A} = 0 .$$

Thus, hodograph should be determined as:

$$\bar r(t) = \bar i \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right) + \bar k \left(h_0 + h_1 t + h_2 t^2 + h_3 t^3 \right),$$

i.e. $\quad r_1(t) = \rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 , \qquad \bar r_3(t) = h_0 + h_1 t + h_2 t^2 + h_3 t^3$

Hence, $\quad \dot r_1(t) = \rho_1 + 2\rho_2 t + 3\rho_3 t^2 \qquad \dot r_3(t) = h_1 + 2h_2 t + 3h_3 t^2 .$

Then according to boundary conditions we obtain:

$$r_{1A}(0) = \rho_0 , \qquad r_{3A}(0) = h_0 ,$$
$$\dot r_{1A}(0) = \rho_1 , \qquad \dot r_{3A}(0) = h_1 ,$$
$$r_{1B}(t_k) = \rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 , \qquad r_{3B}(t_k) = h_0 + h_1 t_k + h_2 t_k^2 + h_3 t_k^3 ,$$
$$\dot r_{1B}(t_k) = \rho_1 + 2\rho_2 t_k + 3\rho_3 t_k^2 , \qquad \dot r_{3B}(t_k) = h_1 + 2h_2 t_k + 3h_3 t_k^2 ,$$

i.e.,

$$\rho_0 = r_{1A}, \qquad h_0 = 0,$$
$$\rho_1 = V_{1A}, \qquad h_1 = 0,$$
$$\rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 = r_{1B}, \qquad h_0 + h_1 t_k + h_2 t_k^2 + h_3 t_k^3 = r_{3B},$$
$$\rho_1 + 2\rho_2 t_k + 3\rho_3 t_k^2 = V_{1B}, \qquad h_1 + 2h_2 t_k + 3h_3 t_k^2 = 0,$$

where

$$t_k = \frac{r_{1B} - r_{1A}}{V_{1A}}, \qquad V_{3cp} = \frac{r_{3B} - r_{3A}}{t_k},$$

as $V_{1A} = V_{1B}$, $V_{3A} = V_{3B} = 0$.

From which we obtain two linear systems of non-homogeneous algebraic equations:

$$\begin{cases} \rho_0 = r_{1A} \\ \rho_1 = V_{1A} \\ \rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 = r_{1B} \\ \rho_1 + 2\rho_2 t_k + 3\rho_3 t_k^2 = V_{1B} \end{cases}; \qquad \begin{cases} h_0 = 0 \\ h_1 = 0 \\ h_0 + h_1 t_k + h_2 t_k^2 + h_3 t_k^3 = r_{3B} \\ h_1 + 2h_2 t_k + 3h_3 t_k^2 = 0 \end{cases},$$

which extended matrices are:

$$\left\| \begin{array}{cccc|c} 1 & 0 & 0 & 0 & r_{1A} \\ 0 & 1 & 0 & 0 & V_{1A} \\ 1 & t_k & t_k^2 & t_k^3 & r_{1B} \\ 0 & 1 & 2t_k & 3t_k^2 & V_{1B} \end{array} \right\|, \qquad \left\| \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & t_k & t_k^2 & t_k^3 & r_{3B} \\ 0 & 1 & 2t_k & 3t_k^2 & 0 \end{array} \right\|.$$

Applying Jordan-Gauss method, we find:

$$\left\| \begin{array}{cccc|c} 1 & 0 & 0 & 0 & r_{1A} \\ 0 & 1 & 0 & 0 & V_{1A} \\ 0 & 0 & 1 & 0 & 3\frac{r_{1B} - r_{1A}}{t_k^2} - \frac{V_{1B} + 2V_{1A}}{t_k} \\ 0 & 0 & 0 & 1 & \frac{V_{1B} + 2V_{1A}}{t_k^2} - 2\frac{r_{1B} - r_{1A}}{t_k^3} \end{array} \right\|; \qquad \left\| \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3\frac{r_{1B}}{t_k^2} \\ 0 & 0 & 0 & 1 & -2\frac{r_{3B}}{t_k^3} \end{array} \right\|,$$

i.e.,

$$\rho_0 = r_{1A}, \qquad h_0 = 0,$$
$$\rho_1 = V_{1A}, \qquad h_1 = 0,$$
$$\rho_2 = 3\frac{r_{1B} - r_{1A}}{t_k^2} - \frac{V_{1B} + 2V_{1A}}{t_k}, \qquad h_2 = 3\frac{r_{3B}}{t_k^2},$$
$$\rho_3 = \frac{V_{1B} + V_{1A}}{t_k^2} - 2\frac{r_{1B} - r_{1A}}{t_k^3}, \qquad h_3 = -2\frac{r_{3B}}{t_k^3},$$

or, taking away t_k and taking into consideration that $V_{1A} = V_{1B}$, we obtain:

$$\rho_0 = r_{1A}, \qquad h_0 = 0,$$
$$\rho_1 = V_{1A}, \qquad h_1 = 0,$$
$$\rho_2 = 0, \qquad h_2 = 3\left(\frac{V_{1A}}{r_{1B} - r_{1A}}\right)^2 r_{3B},$$

$$\rho_3 = 0, \qquad h_3 = -2\left(\frac{V_{1A}}{r_{1B}-r_{1A}}\right)^3 r_{3B}.$$

Hence, in the case under consideration, the highway hodograph is:

$$\bar{r}(t) = \bar{i}\,(r_{1A}+V_{1A}t)+\bar{k}\left[3\left(\frac{V_{1A}}{r_{1B}-r_{1A}}\right)^2 r_{3B}t^2 - 2\left(\frac{V_{1A}}{r_{1B}-r_{1A}}\right)^3 r_{3B}t^3\right],$$

or

$$\bar{r}(t) = \bar{i}\,(r_{1A}+V_{1A}t)+\bar{k}\left(3-2\frac{V_{1A}}{r_{1B}-r_{1A}}t\right)\left(\frac{V_{1A}}{r_{1B}-r_{1A}}\right)^2 r_{3B}t^2.$$

4.2. The highway profile within an area having rise (slope), without turn, and with equal speed at the beginning and the end

Trajectory or a form of the highway lone within specified area in vertical plane of earth coordinate system is shown parametrically:

$$\begin{cases} r_1(t) = r_{1A}+V_{1A}t \\ r_3(t) = 3r_{3B}\left(\dfrac{V_{1A}t}{r_{1B}-r_{1A}}\right)^2 - 2r_{3B}\left(\dfrac{V_{1A}t}{r_{1B}-r_{1A}}\right)^3 \end{cases},$$

where t is time-parameter.

Taking away t parameter $\qquad t = \dfrac{r_1(t)-r_{1A}}{V_{1A}}$

And introducing new variables:

$$x = \frac{r_1(t)-r_{1A}}{r_{1B}-r_{1A}}, \qquad z = \frac{r_3(t)}{r_{3B}},$$

we obtain the highway profile as a superimposition of square parabola and cubic parabola: $z = 3x^2 - 2x^3$.

Where variation range is:	$0 \le x \le 1$,	$0 \le z \le 1$;	
Extremes are:	$x_1^{\text{ý}} = 0$,	$x_2^{\text{ý}} = 1$,	$(z' = 0)$;
Flex point is:	$x_1^{\text{ã}} = \dfrac{1}{2}$,		$(z'' = 0)$;
Concavity interval is:	$0 < x < \dfrac{1}{2}$,		$(z'' > 0)$;
Convexity interval is:	$\dfrac{1}{2} < x < 1$,		$(z'' < 0)$;
Extremes are:	$z_{\min}(0) = 0$,		$z_{\max}(1) = 1$.

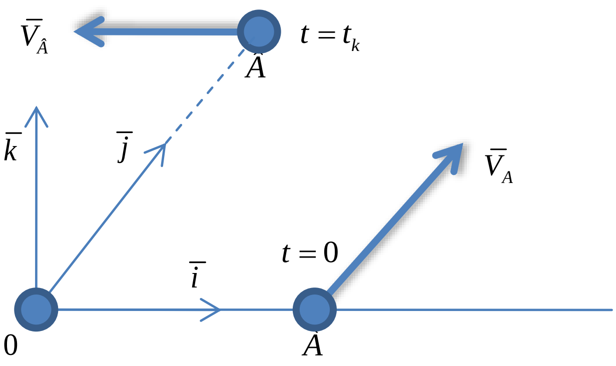
4.3. Construction of hodograph for vehicle movement while turning in horizontal plane

Boundary conditions are:

$$\bar{r}_A = \bar{i}\,r_{1A}, \qquad \bar{r}_B = \bar{j}\,r_{2B},$$

$$\bar{V}_A = \bar{j}\,V_{2A}\,, \quad \bar{V}_B = -\bar{i}\,V_{1B}$$

being shown in the figure:



where $r_{2A} = 0$, $r_{3A} = 0$, $r_{3B} = 0$, $r_{1B} = 0$,
$V_{1A} = 0$, $V_{3A} = 0$, $V_{2B} = 0$, $V_{3B} = 0$.

Turn is performed by a specified angle φ_0. For example, it is assumed that $\varphi_0 = \frac{\pi}{2}$.
Hence, highway hodograph while turning in horizontal plane is determined as:

$$\bar{r}(t) = \begin{Vmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 \end{Vmatrix} \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \left(\bar{i}\cos\omega t + \bar{j}\sin\omega t \right),$$

i.e. varied parameters h_0, h_1, h_2, h_3, are identically equal to zero and ρ_0, ρ_1, ρ_2, ρ_3, parameters are identified according to boundary conditions. In this context, hodograph components are:

$$\bar{r}_1(t) = \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right)\cos\omega t\,,$$

$$\bar{r}_2(t) = \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right)\sin\omega t$$

and respectively:

$$\dot{r}_1(t) = \left(\rho_1 + 2\rho_2 t + 3\rho_3 t^2 \right)\cos\omega t - \omega\left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right)\sin\omega t\,,$$

$$\dot{r}_2(t) = \left(\rho_1 + 2\rho_2 t + 3\rho_3 t^2 \right)\sin\omega t + \omega\left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right)\cos\omega t\,.$$

According to boundary conditions we obtain:

$$r_{1A}(0) = \rho_0\,, \quad r_{2A}(0) = 0\,,$$

$$\dot{r}_{1A}(0) = \rho_1\,, \quad \dot{r}_{2A}(0) = \omega\,\rho_0\,,$$

$$r_{1B}(t_k) = 0\,, \quad \dot{r}_{2B}(t_k) = \rho_1 + 2\rho_2 t + 3\rho_3 t^2$$

or dropping trivial conditions,

$$\rho_0 = r_{1A}\,, \qquad \omega\,\rho_0 = V_{2A}\,,$$

$$\rho_1 = 0\,, \qquad \rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 = r_{2B}\,,$$

$$\omega\left(\rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 \right) = V_{1B}\,, \qquad \rho_1 + 2\rho_2 t_k + 3\rho_3 t_k^2 = 0\,.$$

Hence: $\omega r_{1A} = V_{2A}$, $\frac{V_{1B}}{\omega} = r_{2B}$ or $\omega = \frac{V_{2A}}{r_{1A}}$, $\omega = \frac{V_{1B}}{r_{2B}}$, i.e. $\frac{V_{2A}}{r_{1A}} = \frac{V_{1B}}{r_{2B}}$, and $V_{2A} \cdot r_{2B} = V_{1B} r_{1A}$.

Following equations are independent:

$$\begin{cases} t_k^2 \left(\rho_2 + \rho_3 t_k \right) = r_{2B} - r_{1A} \\ t_k \left(2\rho_2 + 3\rho_3 t_k \right) = 0 \end{cases}.$$

Assuming that: $\omega t_k = \frac{\pi}{2}$, we find $t_k = \frac{\pi}{2\omega}$, and variable parameters are determined by:

$$\rho_0 = r_{1A}, \qquad \rho_1 = 0,$$

$$\rho_2 = 3\frac{r_{2B} - r_{1A}}{t_k^2}, \qquad \rho_3 = -2\frac{r_{2B} - r_{1A}}{t_k^3},$$

or

$$\rho_2 = -\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{2B} - r_{1A} \right), \quad \rho_3 = -\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{2B} - r_{1A} \right),$$

and

$$\rho_2 = -\frac{12}{\pi^2}\left(\frac{V_{1B}}{r_{2B}}\right)^2 \left(r_{2B} - r_{1A} \right), \quad \rho_3 = -\frac{16}{\pi^3}\left(\frac{V_{1B}}{r_{2B}}\right)^3 \left(r_{2B} - r_{1A} \right).$$

Hence, highway hodograph while turning in horizontal plane by right angle is as follows:

$$\bar{r}(t) = \left[r_{1A} + \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{2B} - r_{1A} \right) t^2 - \frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{2B} - r_{1A} \right) t^3 \right] \cdot \left(\bar{i} \cos\frac{V_{2A}}{r_{1A}} t + \bar{j} \sin\frac{V_{2A}}{r_{1A}} t \right),$$

or

$$\bar{r}(t) = \left[r_{1A} + \frac{12}{\pi^2}\left(\frac{V_{1B}}{r_{2B}}\right)^2 \left(r_{2B} - r_{1A} \right) t^2 - \frac{16}{\pi^3}\left(\frac{V_{1B}}{r_{2B}}\right)^3 \left(r_{2B} - r_{1A} \right) t^3 \right] \cdot \left(\bar{i} \cos\frac{V_{1B}}{r_{2B}} t + \bar{j} \sin\frac{V_{1B}}{r_{2B}} t \right).$$

4.4. Highway plan while turning by right angle

In a plan a trajectory or a form of highway line in the Earth-based coordinate system is shown parametrically:

$$r_1(t) = \left[r_{1A} + \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{2B} - r_{1A} \right) t^2 - \frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{2B} - r_{1A} \right) t^3 \right] \cos\frac{V_{2A}}{r_{1A}} t,$$

$$r_2(t) = \left[r_{1A} + \frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2 \left(r_{2B} - r_{1A} \right) t^2 - \frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3 \left(r_{2B} - r_{1A} \right) t^3 \right] \sin\frac{V_{2A}}{r_{1A}} t$$

or, having dropped parameter t (time)with the help of condition $t = \frac{\varphi}{\omega}$,

where φ is polar angle and introducing polar radius $r^2(t) = r_1^2(t) + r_2^2(t)$ we obtain the following in polar coordinate system:

$$\frac{r(\varphi) - r_{1A}}{r_{2B} - r_{1A}} = 3\left(\frac{\varphi}{\pi/2}\right)^2 - 2\left(\frac{\varphi}{\pi/2}\right)^3,$$

i.e. superposition of quadratic and cubic Archimedean spirals.

It is obvious that:

$$\varphi = 0, \qquad r(0) = r_{1A};$$

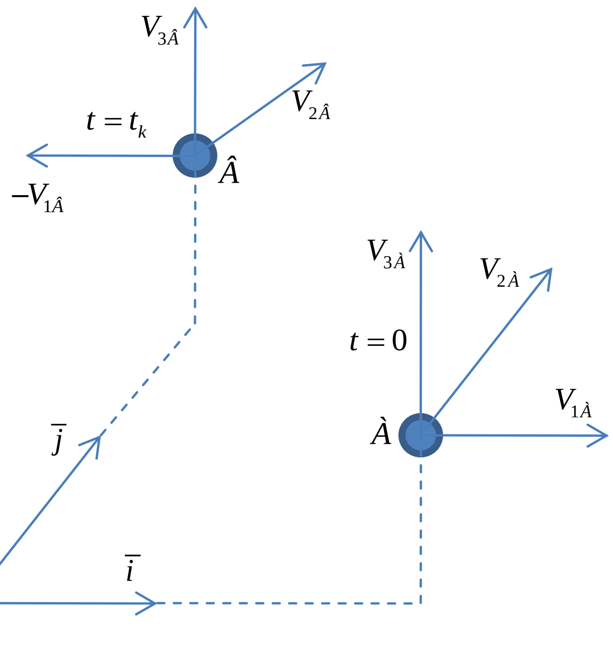
$$\varphi = \frac{\pi}{2}, \qquad r\left(\frac{\pi}{2}\right) = r_{2B};$$

$$\varphi = \frac{\pi}{4}, \qquad r\left(\frac{\pi}{4}\right) = \frac{1}{2}(r_{2B} + r_{1A}).$$

Moreover, when $V_{2A} = V_{1B}$ or $r_{1A} = r_{2B}$ it follows: $r(\varphi) = r_{1A}$ at any φ, i.e. we obtain radial arc within the given interval: $0 \le \varphi \le \frac{\pi}{2}$.

4.5. Construction of hodograph for vehicle movement while rising (sloping) and turning along spatial highway

The figure shows boundary conditions within considered highway area:



being:

$$t = 0, \qquad t = t_k,$$

$$\bar{r}_A = \bar{i}\, r_{1A} + \bar{k}\, r_{3A}, \qquad \bar{r}_B = \bar{i}\, r_{2B} + \bar{k}\, r_{3B},$$

$$\bar{V}_A = \bar{i}\, V_{1A} + \bar{j}\, V_{2A} + \bar{k}\, V_{3A}, \qquad \bar{V}_B = -\bar{i}\, V_{1B} + \bar{j}\, V_{2B} + \bar{k}\, V_{3B}.$$

Where $r_{2A} = 0$, $r_{1B} = 0$.

We also suppose that $\omega t_k = \frac{\pi}{2}$, i.e. turn is performed by a right angle. Then hodograph of vehicle movement while rising $(r_{3B} - r_{3A} > 0)$ and turning by the specified angle $\left(\varphi_0 = \frac{\pi}{2}\right)$ along spatial highway should be found as follows:

$$\bar{r}(t) = \begin{Vmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 \end{Vmatrix} \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} \left(\bar{i} \cos \omega t + \bar{j} \sin \omega t \right) + \begin{Vmatrix} h_0 & h_1 & h_2 & h_3 \end{Vmatrix} \begin{Vmatrix} 1 \\ t \\ t^2 \\ t^3 \end{Vmatrix} .$$

In this context hodograph components are:

$$r_1(t) = \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right) \cos \omega t ,$$
$$r_2(t) = \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right) \sin \omega t ,$$
$$r_3(t) = h_0 + h_1 t + h_2 t^2 + h_3 t^3 ,$$

and components of speed vectors are respectively:

$$\dot{r}_1(t) = \left(\rho_1 + 2\rho_2 t + 3\rho_3 t^2 \right) \cos \omega t - \omega \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right) \sin \omega t ,$$
$$\dot{r}_2(t) = \left(\rho_1 + 2\rho_2 t + 3\rho_3 t^2 \right) \sin \omega t + \omega \left(\rho_0 + \rho_1 t + \rho_2 t^2 + \rho_3 t^3 \right) \cos \omega t ,$$
$$\dot{r}_3(t) = h_1 + 2h_2 t + 3h_3 t^2 ,$$

where $\rho_0,\ \rho_1,\ \rho_2,\ \rho_3$ and $h_0,\ h_1,\ h_2,\ h_3$ as well as ω are variable hodograph parameters. According to the given boundary conditions we have for a point $A(t=0)$:

$$r_{1A}(0) = \rho_0 , \qquad r_{2A}(0) = 0 , \qquad r_{3A}(0) = h_0 ,$$
$$\dot{r}_{1A}(0) = \rho_1 , \qquad \dot{r}_{2A}(0) = \omega\, \rho_0 , \qquad \dot{r}_{3A}(0) = h_1 ;$$

for a point $B(t = t_k)$:

$$r_{1B}(t_k) = 0 , \qquad r_{2B}(t_k) = \rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 , \qquad r_{3B}(t_k) = h_0 + h_1 t_k + h_2 t_k^2 + h_3 t_k^3 ,$$
$$\dot{r}_{1B}(t_k) = -\omega \left(\rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 \right), \quad \dot{r}_{2B}(t_k) = \rho_1 + 2\rho_2 t_k + 3\rho_3 t_k^2 , \quad \dot{r}_{3B}(t_k) = h_1 + 2h_2 t_k + 3h_3 t_k^2 .$$

Hence:

$$\rho_0 = r_{1A} , \qquad \rho_1 = V_{1A} , \qquad \omega r_{1A} = V_{2A} , \qquad h_0 = r_{3A} , \qquad h_1 = V_{3A} ;$$
$$\rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 = r_{2B} ,$$
$$h_0 + h_1 t_k + h_2 t_k^2 + h_3 t_k^3 = r_{3B} ,$$
$$\omega \left(\rho_0 + \rho_1 t_k + \rho_2 t_k^2 + \rho_3 t_k^3 \right) = V_{1B} ,$$
$$h_1 + 2h_2 t_k + 3h_3 t_k^2 = V_{3B} ,$$
$$\rho_1 + 2\rho_2 t_k + 3\rho_3 t_k^2 = V_{2B} .$$

It is obvious that $\omega = \dfrac{V_{2A}}{r_{1A}}$ or $\omega = \dfrac{V_{1B}}{r_{2B}}$,

i.e.

$$\frac{V_{2A}}{V_{1B}} = \frac{r_{1A}}{r_{2B}} ,$$

and

$$t_k = \frac{\pi}{2\omega} \quad \text{or} \quad t_k = \frac{\pi \cdot r_{1A}}{2V_{2A}} , \qquad t_k = \frac{\pi \cdot r_{2B}}{2V_{1B}} .$$

Then following equation systems are independent:

$$\begin{cases} t_k^2\left(\rho_2+\rho_3 t_k\right)=r_{2B}-r_{1A}-V_{1A}\cdot\dfrac{\pi\cdot r_{1A}}{2V_{2A}} \\ t_k\left(2\rho_2+3\rho_3 t_k\right)=V_{2B}-V_{1A} \end{cases},$$

$$\begin{cases} t_k^2\left(h_2+h_3 t_k\right)=r_{3B}-r_{3A}-V_{3A}\cdot\dfrac{\pi\cdot r_{1A}}{2V_{2A}} \\ t_k\left(2h_2+3h_3 t_k\right)=V_{3B}-V_{3A} \end{cases},$$

and their solutions are:

$$\rho_2=3\frac{r_{2B}-r_{1A}}{t_k^2}-\frac{V_{2B}+2V_{1A}}{t_k}\ ,\quad \rho_3=-2\frac{r_{2B}-r_{1A}}{t_k^3}+\frac{V_{2B}+V_{1A}}{t_k^2}\ ,$$

$$h_2=3\frac{r_{3B}-r_{3A}}{t_k^2}-\frac{V_{3B}+2V_{3A}}{t_k}\ ,\quad h_3=-2\frac{r_{3B}-r_{3A}}{t_k^3}+\frac{V_{3B}+V_{3A}}{t_k^2}\ ,$$

or using these formulas to describe time for vehicle to pass the specified highway area (t_k) through kinematic initial data in boundary points of the area we obtain:

$$\rho_2=\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2\left(r_{2B}-r_{1A}\right)-\frac{2}{\pi}\cdot\frac{V_{2A}}{r_{1A}}\left(V_{2B}+2V_{1A}\right)\ ,$$

$$\rho_3=-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3\left(r_{2B}-r_{1A}\right)+\frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2\left(V_{2B}+V_{1A}\right)\ ,$$

$$h_2=\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2\left(r_{3B}-r_{3A}\right)-\frac{2}{\pi}\cdot\frac{V_{2A}}{r_{1A}}\left(V_{3B}+2V_{3A}\right)\ ,$$

$$h_3=-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3\left(r_{3B}-r_{3A}\right)+\frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2\left(V_{3B}+V_{3A}\right)\ .$$

Thus, hodograph of vehicle movement while rising (sloping) and turning by a right angle along spatial highway is determined as follows:

$$\begin{aligned}\bar r\left(t\right)=&\left[r_{1A}+V_{1A}t+\frac{2}{\pi}\frac{V_{2A}}{r_{1A}}\left(\frac{6}{\pi}\cdot\frac{V_{2A}}{r_{1A}}\left(r_{2B}-r_{1A}\right)-V_{2B}-2V_{1A}\right)t^2+\right.\\&\left.+\frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2\left(V_{2B}+V_{1A}-\frac{4}{\pi}\frac{V_{2A}}{r_{1A}}\left(r_{2B}-r_{1A}\right)\right)t^3\right]\cdot\left(\bar i\cos\frac{V_{2A}}{r_{1A}}t+\bar j\sin\frac{V_{2A}}{r_{1A}}t\right)+\\&+\left[r_{3A}+V_{3A}t+\frac{2}{\pi}\frac{V_{2A}}{r_{1A}}\left(\frac{6}{\pi}\cdot\frac{V_{2A}}{r_{1A}}\left(r_{3B}-r_{3A}\right)-V_{3B}-2V_{3A}\right)t^2+\right.\\&\left.+\frac{4}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2\left(V_{3B}+V_{3A}-\frac{4}{\pi}\frac{V_{2A}}{r_{1A}}\left(r_{3B}-r_{3A}\right)\right)t^3\right]\cdot\bar k.\end{aligned}$$

In particular, when $V_{1A}=0$, $V_{3A}=0$, $V_{2B}=0, V_{3B}=0$, we obtain:

$$\bar{r}(t)=\left[r_{1A}+\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2(r_{2B}-r_{1A})t^2-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3(r_{2B}-r_{1A})t^3\right]\cdot\left(\bar{i}\cos\frac{V_{2A}}{r_{1A}}t+\bar{j}\sin\frac{V_{2A}}{r_{1A}}t\right)+$$
$$+\left[r_{3A}+\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2(r_{3B}-r_{3A})t^2-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3(r_{3B}-r_{3A})t^3\right]\cdot\bar{k}.$$

Then, supposing that $r_{3A}=0$, $r_{3B}=0$, we obtain previously considered hodograph of vehicle movement while turning by a right angle within horizontal plane.

4.6. Highway spatial trajectory while rising (sloping) and turning by a right angle

Parametrically the trajectory is:

$$r_1(t)=\left[r_{1A}+\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2(r_{2B}-r_{1A})t^2-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3(r_{2B}-r_{1A})t^3\right]\cos\frac{V_{2A}}{r_{1A}}t,$$
$$r_2(t)=\left[r_{1A}+\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2(r_{2B}-r_{1A})t^2-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3(r_{2B}-r_{1A})t^3\right]\sin\frac{V_{2A}}{r_{1A}}t,$$
$$r_3(t)=r_{3A}+\frac{12}{\pi^2}\left(\frac{V_{2A}}{r_{1A}}\right)^2(r_{3B}-r_{3A})t^2-\frac{16}{\pi^3}\left(\frac{V_{2A}}{r_{1A}}\right)^3(r_{3B}-r_{3A})t^3.$$

Shifting from Cartesian coordinate system

$$x(t)=r_1(t),\quad y(t)=r_2(t),\quad z(t)=r_3(t)$$

to cylindrical one

$$r^2(t)=x^2(t)+y^2(t),\; z(t)=r_3(t)$$

and introducing polar angle as an argument

$$\varphi=\frac{V_{2A}}{r_{1A}}t\ ,$$

we transform initial trajectory as follows:

$$\frac{r(y)-r_{1A}}{r_{2B}-r_{1A}}=3\left(\frac{\varphi}{\pi/2}\right)^2-2\left(\frac{\varphi}{\pi/2}\right)^3;\qquad \frac{z(y)-r_{3A}}{r_{3B}-r_{3A}}=3\left(\frac{\varphi}{\pi/2}\right)^2-2\left(\frac{\varphi}{\pi/2}\right)^3.$$

Hence, dropping polar angle we determine linear dependence of the required variables:

$$\frac{z-r_{3A}}{r_{3B}-r_{3A}}=\frac{r-r_{1A}}{r_{2B}-r_{1A}},$$

where variation range is determined by boundary conditions:

$$r_{3A}\le z\le r_{3B},\; r_{1A}\le r\le r_{2B}.$$

CONCLUSION

Consequently, hodograph of vehicle movement depending upon various modes – accelerated, decelerated, with constant speed, injunctions and turns, within straight and curved areas, while sloping and rising can be synthesized in a class of proposed spiral lines (guidepaths) corresponding to true, undisturbed trajectories of a vehicle considered as a material point in this paper.

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