ON REGULARITY OF WEAK SOLUTIONS TO ONE CLASS OF INITIAL-BOUNDARY VALUE PROBLEM WITH PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. We discuss solvability and some extra regularity properties for the weak solutions to one class of the initial-boundary value problem arising in the study of the dynamics of an arterial system.

Key words. weak solution, Boussinesq system, existence result, regularity properties.

AMS subject classifications. 49J20

1. Introduction. It is well known that the cardiovascular system transports oxygen and nutrients to all the tissues of the body, from where it removes carbon dioxide and other harmful waste products of cell metabolism. From a physical point of view, the system consists of a pump that propels a viscous liquid (the blood) through a network of flexible tubes. The heart provides energy to move blood through the circulatory system and is one key component in the complex control mechanism of maintaining pressure in the vascular system [18]. The aorta is the main artery originating from the left ventricle and then bifurcates to other arteries, and is identified by several segments (ascending, thoracic, abdominal). There are several features of the aorta that have an effect on the blood flow, such as the tapering of the aorta or the fact that ascending aorta is arched (curved). Still, the functionality of the aorta, considered as a single segment, is worth exploring from a modeling perspective, in particular in relationship to the presence of the aortic valve.

There has been extensive literature describing the dynamics of the vascular network coupled with a heart model (e.g. [8], [9], [10], [11], [17], [21]), the majority focusing on either a detailed description of the four chambers of the heart or on the spatial dynamics in the aorta, but not on both. In fact, there seem to be no studies addressing the heart rate variability based on the detailed spatial description of the pressure and flow patterns in the aorta.

Taking into account the elasticity of the aorta, considered as a single vessel, together with an aortic valve model at the inflow and a peripheral resistance model at the outflow, we can capture through simulation the dynamics of the pressure and flow in the aorta as well as the heart rate variability. In view of this, we make use of the standard viscous hyperbolic system (see [2], [15], [21]) which models cross-section area $S(x,t)$ and average velocity $u(x,t)$ in the spatial domain:

$$\frac{\partial S}{\partial t} + \frac{\partial (Su)}{\partial x} - \nu \frac{\partial^2 S}{\partial x^2} = 0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = f, \tag{1.2}$$

where $(t,x) \in Q = (0,T) \times (0,L)$, $f = f(x,t)$ is a friction force, usually taken to be $f = -22\mu\pi u/S$, μ is the fluid viscosity, $P(x,t)$ is the hydrodynamic pressure, L is the length of an arterial segment, and $T = T_{pulse} = 60/(HartRate)$ is the duration of an entire

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heartbeat. Here we include the inertial effects of the wall motion, described by the wall displacement $\eta = \eta(x,t)$:

$$\eta = r - r_0 = \frac{1}{\sqrt{\pi}}(\sqrt{S} - \sqrt{S_0}) \simeq \frac{S - S_0}{2\sqrt{\pi S_0}}. \tag{1.3}$$

The fluid structure interaction is modeled using inertial forces, which gives the pressure law (see [3], [8])

$$P = P_{ext} + \frac{\beta}{r_0^2}\eta + \rho_\omega h \frac{\partial^2 \eta}{\partial t^2} = P_{ext} + \frac{\beta}{S_0}(\sqrt{S} - \sqrt{S_0}) + m\frac{\partial^2 S}{\partial t^2}, \tag{1.4}$$

where $r(x,t)$ is the radius, $r_0 = r(x,0)$, $S_0 = S(x,0)$, P_{ext} is the external pressure, $\beta = \frac{E}{1-\sigma^2}h$, σ is the Poisson ratio (usually taken to be $\sigma^2 = \frac{1}{2}$), E is Young modulus, h is the wall thickness, $m = \frac{\rho_\omega h}{2\sqrt{\pi A_0}}$, ρ_ω is the density of the wall.

This leads to the following Boussinesq system (for the details we refer to [4]):

$$\begin{cases} \eta_t + \eta_x u + \dfrac{1}{2}(\eta + r_0)u_x = 0, \\ u_t + uu_x + \dfrac{2Eh}{\rho r_0^2}\eta_x + \dfrac{\rho_\omega h}{\rho}\eta_{xtt} = f, \end{cases} \tag{1.5}$$

where ρ is the blood density. Considering the relation $\eta_t = -\frac{1}{2}r_0 u_x$, we get the system:

$$\begin{cases} \eta_t + \eta_x u + \dfrac{1}{2}(\eta + r_0)u_x = 0, \\ u_t + uu_x + \dfrac{2Eh}{\rho r_0^2}\eta_x - \dfrac{\rho_\omega h r_0}{2\rho}u_{xxt} = f, \end{cases} \tag{1.6}$$

or, rearranging terms in u,

$$\begin{cases} \eta_t + \eta_x u + \dfrac{1}{2}(\eta + r_0)u_x = 0, \\ \left(u - \dfrac{\rho_\omega h r_0}{2\rho}u_{xx}\right)_t + \dfrac{1}{2}(u^2)_x + \dfrac{2Eh}{\rho r_0^2}\eta_x = f. \end{cases} \tag{1.7}$$

It remains to furnish the system (1.7) by corresponding initial and boundary conditions.

Since the solvability of the corresponding initial-boundary value problem is not clear in the case of non-homogeneous Dirichlet boundary conditions, the aim of this paper is consider a relaxed statement of this problem. Namely, following the method of vanishing viscosity, we suppose that $\nu\eta_{xx}$ is small enough, and as a result, instead of (1.7), we can deal with the nonlinear problem of Sobolev type.

2. Preliminaries. Let $T > 0$ be a given value. Let also $\Omega = (0,1)$, $Q = (0,T) \times \Omega$, and $\Sigma = (0,T) \times \partial\Omega$. Let $\delta : \mathbb{R} \to \mathbb{R}$ be a locally integrable function on $\mathbb{R}$ such that $\delta(x) \geq \delta_0 > 0$ for a.e. $x \in \Omega$. We will use the standard notion $L^2(\Omega, \delta\, dx)$ for the set of measurable functions u on Ω such that

$$\|u\|_{L^2(\Omega,\delta\, dx)} = \left(\int_\Omega u^2 \delta\, dx\right)^{1/2} < +\infty.$$

We set $H = L^2(\Omega)$, $V_0 = H_0^1(\Omega)$, and identify the Hilbert space H with its dual H^*. On H we use the common natural inner product $(\cdot,\cdot)_H$, and endow the Hilbert space V_0 with the inner product

$$(\varphi,\psi)_{V_0} = (\varphi',\psi')_H \quad \text{for all} \quad \varphi,\psi \in V_0.$$

We also make use of the weighted Sobolev space V_δ as the set of functions $u \in V_0$ for which the norm

$$\|u\|_\delta = \left(\int_\Omega \left[u^2 + \delta(u')^2\right]\,dx\right)^{1/2}$$

is finite. Note that due to the estimate

$$\|u\|_{V_0}^2 := \int_\Omega (u')^2\,dx \le \delta_0^{-1}\int_\Omega \delta(u')^2\,dx \le \delta_0^{-1}\int_\Omega \left[u^2+\delta(u')^2\right]\,dx = \delta_0^{-1}\|u\|_{V_\delta}^2 \tag{2.1}$$

the space V_δ is complete with respect to the norm $\|\cdot\|_{V_\delta}$.

We recall that the dual space of the weighted Sobolev space V_δ is equivalent to $V_\delta^* = W^{-1,2}(\Omega,\delta^{-1}\,dx)$ (for more details see [6]).

Remark 2.1. *In what follows, we make use of the following result: if there exists a value $\nu \in [1,+\infty)$ such that $\delta^{-\nu} \in L^1(\Omega)$, then the expression (see [6, pp.46]):*

$$\|y\|_{V_\delta} = \left[\int_\Omega (u')^2\delta\,dx\right]^{1/2} \tag{2.2}$$

can be considered as a norm on V_δ and it is equivalent to the norm $\|\cdot\|_\delta$. Moreover, in this case the embedding $V_\delta \hookrightarrow L^2(\Omega)$ is compact. Since

$$\|\delta^{-1}\|_{L^1(\Omega)} = \int_\Omega |\delta^{-1}|\,dx \le \delta_0^{-1}|\Omega| < +\infty,$$

it follows that $\nu = 1$ satisfies the inclusion $\nu \in [1,+\infty)$.

Recall that V_0 and, hence, V_δ are continuously embedded into $C(\overline{\Omega})$, see [1, 14] for instance. Moreover, in view of Friedrich's inequality

$$\|u\|_H \le \|u_x\|_H = \|u\|_{V_0}, \quad \forall\, u \in V_0 \tag{2.3}$$

and the obvious relation, for any $x, y \in \Omega$, $y > x$,

$$u^2(y) = \left(u(x) + \int_x^y u'(s)\,ds\right)^2 \le \left(u(x) + \sqrt{y-x}\,\|u'\|_H\right)^2 \le 2u^2(x) + 2\|u'\|_H^2,$$

we have

$$\begin{aligned} u^2(y) &= \int_\Omega u^2(y)\,dx \le 2\left(\|u\|_H^2 + \|u'\|_H^2\right) \\ &= 2\left(\|u\|_H^2 + \|u\|_{V_0}^2\right) = 4\|u\|_{V_0}^2, \quad \forall\, y \in \Omega. \end{aligned}$$

Therefore,

$$\|u\|_{L^\infty(\Omega)} \le 2\|u\|_{V_0} \quad \forall\, u \in V_0. \tag{2.4}$$

We also recall the Agmon's inequality (see [22, p.52]): there exists a constant $C_A > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \le C_A \|u\|_H^{1/2} \|u\|_{V_0}^{1/2} \quad \forall\, u \in V_0. \tag{2.5}$$

Remark 2.2. *Since $\delta, \delta^{-1} \in L^1(\Omega)$, it follows that V_δ is a uniformly convex separable Banach space [14]. Moreover, in view of the estimate (2.1), the embedding $V_\delta \hookrightarrow H$ is continuous and dense. Hence, $H = H^*$ is densely and continuous embedded in V_δ^*, and, therefore, $V_\delta \hookrightarrow H \hookrightarrow V_\delta^*$ is a Hilbert triplet (see [12] for the details).*

By $L^2(0,T;V_0)$ we denote the space of (equivalence classes) of measurable abstract functions $u : [0,T] \to V_0$ such that

$$\|u\|_{L^2(0,T;V_0)} := \left(\int_0^T \|u(t)\|_{V_0}^2\, dt\| \right)^{1/2} < +\infty.$$

By analogy we can define the spaces $L^2(0,T;V_\delta)$, $L^\infty(0,T;H)$, $L^\infty(0,T;V_\delta)$, and $C([0,T];H)$ (for the details, we refer to [5]). In what follows, when t is fixed, the expression $u(t)$ stands for the function $u(t,\cdot)$ considered as a function in Ω with values into a suitable functional space. When we adopt this convention, we write $u(t)$ instead of $u(t,x)$ and $\dot{u}$ instead of u_t for the weak derivative of u in the sense of distribution

$$\int_0^T \varphi(t) \left\langle \dot{u}(t), v \right\rangle_{V_0^*;V_0} dt = -\int_0^T \dot{\varphi}(t) \left\langle u(t), v \right\rangle_{V_0^*;V_0} dt \quad \forall\, v \in V_0,$$

where $\langle \cdot, \cdot \rangle_{V_0^*;V_0}$ denotes the pairing between V_0^*) and V_0. Here, $V_0^* = H^{-1}(\Omega)$ is the dual space to V_0.

We also make use of the Hilbert spaces

$$W_0(0,T) = \left\{ u \in L^2(0,T;V_0) \ : \ \dot{u} \in L^2(0,T;V_0^*) \right\}$$

and

$$W_\delta(0,T) = \left\{ u \in L^2(0,T;V_\delta) \ : \ \dot{u} \in L^2(0,T;V_\delta^*) \right\},$$

supplied with their common inner product, see [5, p.473], for instance.

Remark 2.3. *The following result is fundamental (see [5]): Let (V_0, H, V_0^*) be a Hilbert triplet, $V_0 \hookrightarrow H \hookrightarrow V_0^*$, with V_0 separable, and let $u \in L^2(0,T;V_0)$ and $\dot{u} \in L^2(0,T;V_0^*)$. Then*

- *$u \in C([0,T];H)$ and there exists $C_E > 0$ such that*

$$\max_{1 \le t \le T} \|u(t)\|_H \le C_E \left[\|u\|_{L^2(0,T;V_0)} + \|\dot{u}\|_{L^2(0,T;V_0^*)} \right]; \tag{2.6}$$

- *if $v \in L^2(0,T;V_0)$ and $\dot{v} \in L^2(0,T;V_0^*)$, then the following integration by parts formula holds:*

$$\int_s^t \left[\left\langle \dot{u}(\gamma), v(\gamma) \right\rangle_{V_0^*;V_0} + \left\langle u(\gamma), \dot{v}(\gamma) \right\rangle_{V_0^*;V_0} \right] d\gamma = (u(t), v(t))_H - (u(s), v(s))_H \tag{2.7}$$

for all $s, t \in [0,T]$.

Moreover, as immediately follows from Remark 2.2, the similar assertion is valid for the Hilbert triplet $V_\delta \hookrightarrow H \hookrightarrow V_\delta^*$*.*

In what follows, we make use of the following technical result (see (2.7) for comparison)

Lemma 2.1. *Let* $u \in W_\delta(0,T)$ *be a given distribution and let* $a_\delta : V_\delta \times V_\delta \to \mathbb{R}$ *be the bilinear form which is defined as follows*

$$a_\delta(u,v) = \int_\Omega \delta u' v'\,dx, \quad \forall\, u,v \in V_\delta. \tag{2.8}$$

Then

$$2\int_s^t \Big[\langle \dot u(\gamma), u(\gamma)\rangle_{V_\delta^*;V_\delta} + a_\delta\left(\dot u(\gamma), u(\gamma)\right)\Big]\,d\gamma \tag{2.9}$$

$$= \|u(t)\|_H^2 + \|u(t)\|_{V_\delta}^2 - \|u(s)\|_H^2 - \|u(s)\|_{V_\delta}^2 \quad \textit{for all } s,t \in [0,T]. \tag{2.10}$$

Proof. We set

$$\widehat{u}(t) = \begin{cases} u(t), & t \in [0,T], \\ 0, & \text{otherwise} \end{cases}$$

and regularize it by the convolution in t, i.e. we consider

$$u_\varepsilon = \widehat{u} * \rho_\varepsilon, \quad \text{ where } \ \rho_\varepsilon(t) = \frac{1}{\varepsilon}\rho\left(\frac{t}{\varepsilon}\right),\ \rho \in \mathcal{D}_+(\mathbb{R}),\ \int_{\mathbb{R}} \rho(t)\,dt = 1.$$

As a result, we obtain a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ with the properties

$$\begin{cases} u_\varepsilon \in C^\infty([0,T];V_\delta), \quad \forall\,\varepsilon>0, \\ u_\varepsilon \to u \quad \text{strongly in } L^2_{loc}(0,T;V_\delta) \ \text{ as } \varepsilon \to 0, \\ \dot u_\varepsilon \to \dot u \quad \text{strongly in } L^2_{loc}(0,T;V_\delta), \ \text{ as } \varepsilon \to 0. \end{cases} \tag{2.11}$$

It is easy to see that for each $\varepsilon > 0$ the following equalities

$$\frac{d}{dt}\Big[\underbrace{\left(u_\varepsilon(t), u_\varepsilon(t)\right)_H + a_\delta\left(u_\varepsilon(t), u_\varepsilon(t)\right)}_{\|u_\varepsilon(t)\|_H^2 + \|u_\varepsilon(t)\|_{V_\delta}^2}\Big] = 2\left(\dot u_\varepsilon(t), u_\varepsilon(t)\right)_H + 2\,a_\delta\left(\dot u_\varepsilon(t), u_\varepsilon(t)\right), \tag{2.12}$$

$$\left(\dot u_\varepsilon(t), u_\varepsilon(t)\right)_H = \langle \dot u_\varepsilon(t), u_\varepsilon(t)\rangle_{V_\delta^*;V_\delta}$$

hold true. Moreover, it is worth to note that by properties (2.11) we have:

$$\begin{gathered} \|u_\varepsilon(t)\|_H^2 \to \|u(t)\|_H^2, \\ \left(\dot u_\varepsilon(t), u_\varepsilon(t)\right)_H \to \left(\dot u(t), u(t)\right)_H, \\ a_\delta\left(\dot u_\varepsilon(t), u_\varepsilon(t)\right) \to a_\delta\left(\dot u(t), u(t)\right) \end{gathered}$$

strongly in $L^1_{loc}(0,T)$ as $\varepsilon \to 0$. Taking this fact into account, we can pass to the limit in (2.12) (in the sense of distribution $\mathcal{D}'(0,T)$) as $\varepsilon \to 0$. As a result, we arrive at the relation

$$\frac{d}{dt}\left[\left(u(t),u(t)\right)_H + a_\delta\left(u(t),u(t)\right)\right] = 2\,\langle \dot u(t), u(t)\rangle_{V_\delta^*;V_\delta} + 2a_\delta\left(\dot u(t), u(t)\right) \quad \text{ in } \mathcal{D}'(0,T). \tag{2.13}$$

Since $\|u(\cdot)\|_H^2 \in L^1(0,T)$, $a_\delta\left(u(\cdot),u(\cdot)\right) \in L^1(0,T)$, and $\langle \dot u(\cdot), u(\cdot)\rangle_{V_\delta^*;V_\delta} \in L^1(0,T)$, after integration of (2.13), we arrive at the desired equality (2.9). □

3. Setting of the Dirichlet Initial-Boundary Value Problem. Let $\nu > 0$ be a viscosity parameter, and let

$$f \in L^\infty(0,T;L^2(\Omega)),\ \mu \in L^\infty(0,T;L^2(\Omega)),\ g \in W_0^{1,\infty}(0,T),\ h \in W_0^{1,\infty}(0,T), \tag{3.1}$$

$$u_0 \in V_\delta,\ \eta_0 \in L^\infty(\Omega),\ r_0 \in L^2(\Omega),\ \delta \in L^1(\Omega) \tag{3.2}$$

are given distributions, where f stands for a fixed forcing term, u_0 and η_0 are given initial states, and δ is a singular (probably locally unbounded) weight function such that $\delta(x) \geq \delta_0 > 0$ for a.e. $x \in \Omega$.

The Dirichlet initial-boundary value problem we consider in this paper can be represented in the form of the following viscous Boussinesq system:

$$\begin{cases} \eta_t + \eta_x u + \dfrac{1}{2}\eta u_x + \dfrac{1}{2} r_0 u_x - \nu \eta_{xx} = 0 \quad \text{in } Q, \\ \left[u - (\delta u_x)_x\right]_t + \dfrac{1}{2}(u_x)^2 + \mu \eta_x = f \quad \text{in } Q, \end{cases} \tag{3.3}$$

with the initial

$$\eta(0,\cdot) = \eta_0 \quad u(0,\cdot) = u_0 \quad \text{in } \Omega, \tag{3.4}$$

and boundary conditions

$$\begin{cases} \eta(\cdot,0) = \eta(\cdot,1) = \eta^* \quad \text{in } (0,T), \\ u(\cdot,0) = g(\cdot), \quad u(\cdot,1) = h(\cdot) \quad \text{in } (0,T). \end{cases} \tag{3.5}$$

In order to give a precise description of the weak solutions to this problem, we define the following bilinear and trilinear forms

$$a_1(\varphi,\psi) = \nu \int_\Omega \varphi'\psi'\,dx \quad \forall\, \varphi,\psi \in V_0, \tag{3.6}$$

$$a_2(\varphi,\psi) = \int_\Omega \delta\varphi'\psi'\,dx \quad \forall\, \varphi,\psi \in V_\delta, \tag{3.7}$$

$$b_1(\varphi,\psi,\phi) = \int_\Omega \left[\varphi'\psi\phi + \frac{1}{2}\varphi\psi'\phi\right] dx \quad \forall\, \varphi,\psi,\phi \in V_0, \tag{3.8}$$

$$b_2(\varphi,\psi,\phi) = \frac{1}{3}\int_\Omega \left[(\varphi\psi)'\,\phi + \varphi\psi'\phi\right] dx \quad \forall\, \varphi,\psi,\phi \in V_\delta. \tag{3.9}$$

Since V_δ is continuously embedded into $C(\overline{\Omega})$, it easily follows from (3.6)–(3.9) that each of these forms are continuous. Indeed, let us consider the form $b_1(\varphi,\psi,\phi)$ for instance. We have

$$\begin{aligned} |b_1(\varphi,\psi,\phi)| &\leq \|\phi\|_{C(\overline{\Omega})} \left[\int_\Omega |\varphi'||\psi|\,dx + \frac{1}{2}\int_\Omega |\varphi||\psi'|\,dx\right] \\ &\overset{\text{by (2.4)}}{\leq} 2\|\phi\|_{V_0}\left[\|\varphi'\|_H\|\psi\|_H + \frac{1}{2}\|\varphi\|_H\|\psi'\|_H\right] \overset{\text{by (2.3)}}{\leq} 3\|\phi\|_{V_0}\|\varphi\|_{V_0}\|\psi\|_{V_0}. \end{aligned}$$

Moreover, direct calculations show that

$$b_1(\varphi,\psi,\varphi) = 0 \quad \text{for all } \varphi \in V_0 \text{ and } \psi \in V_\delta, \tag{3.10}$$

$$b_2(\varphi,\varphi,\phi) = \int_\Omega \varphi\varphi'\phi\,dx \quad \text{for all } \varphi,\phi \in V_\delta. \tag{3.11}$$

Definition 3.1. *We say that, for given* $g \in W_0^{1,\infty}(0,T)$ *and* $h \in W_0^{1,\infty}(0,T)$*, a couple of functions* $(\eta(t), u(t))$ *is a weak solution to the initial-boundary value problem* (3.3)–(3.5) *if*

$$\eta(t) = w(t) + \eta^*, \quad u(t) = v(t) + u^*(t), \quad u^*(t) = g(t) - [h(t) - g(t)]\, x, \tag{3.12}$$

$$w(\cdot) \in W_0(0,T), \quad v(\cdot) \in W_\delta(0,T), \tag{3.13}$$

$$(w(0), \chi)_H = (\eta_0 - \eta^*, \chi)_H \quad \textit{for all } \chi \in H, \tag{3.14}$$

$$(u(0), \chi)_{V_\delta} = (u_0, \chi)_{V_\delta} \quad \textit{for all } \chi \in V_\delta, \tag{3.15}$$

$$\langle \dot{w}(t), \varphi \rangle_{V_0^*; V_0} + a_1(w(t), \varphi) + b_1(w(t), v(t), \varphi) + b_1(w(t), u^*(t), \varphi) \tag{3.16}$$

$$+ \frac{1}{2}\left([r_0 + \eta^*]\, v_x(t), \varphi\right)_H + \frac{1}{2}\left([r_0 + \eta^*]\, [g(t) - h(t)], \varphi\right)_H = 0, \tag{3.17}$$

$$\langle \dot{v}(t), \psi \rangle_{V_\delta^*; V_\delta} + a_2(\dot{v}(t), \psi) + b_2(v(t), v(t), \psi) + (v_x(t) u_x^*, \psi)_H + (\mu(t) w_x(t), \psi)_H$$

$$= (f(t), \psi)_H - (\dot{u}^*(t), \psi)_H - b_2(u^*(t), u^*(t), \psi) \tag{3.18}$$

for all $\varphi \in V_0$ *and* $\psi \in V_\delta$ *and a.e.* $t \in [0,T]$.

Remark 3.1. *Let us mention that if we multiply the left- and right-hand sides of equations* (3.17)–(3.18) *by function* $\chi \in L^2(0,T)$ *and integrate the result over the interval* $(0,T)$*, all integrals are finite. Moreover, closely following the arguments of Korpusov and Sveshnikov (see [13]), it can be shown that the weak solution to* (3.3)–(3.5) *in the sense of Definition 3.1 is equivalent to the following one:* $(\eta(t), u(t))$ *is a weak solution to the initial-boundary value problem* (3.3)–(3.5) *if the conditions* (3.12)-(3.15) *hold true and*

$$\int_0^T \langle A_1(w(t), u(t)), \varphi(t) \rangle_{V_0^*; V_0}\, dt = 0, \quad \forall\, \varphi(\cdot) \in L^2(0,T; V_0), \tag{3.19}$$

$$\int_0^T \langle A_2(w(t), u(t)), \psi(t) \rangle_{V_\delta^*; V_\delta}\, dt = 0, \quad \forall\, \psi(\cdot) \in L^2(0,T; V_\delta), \tag{3.20}$$

where

$$A_1(w,u) = \frac{\partial}{\partial t} w - \nu w_{xx} + w_x (v + u^*) + \frac{1}{2}(w + \eta^*)(v_x + u_x^*) + \frac{1}{2} r_0 (v_x + u_x^*), \tag{3.21}$$

$$A_2(w,u) = \frac{\partial}{\partial t}\left(v - (\delta v_x)_x\right) + \frac{1}{2}\left(v^2\right)_x + v_x u_x^* + \mu w_x - f + \frac{\partial}{\partial t} u^* + \frac{1}{2}\left((u^*)^2\right)_x. \tag{3.22}$$

4. On Uniqueness of Weak Solutions to the Viscous Boussinesq System. Let

$$(\eta^i(t), u^i(t)) = \left(w^i(t) + \eta^*, v^i(t) + u^*(t)\right) \in [W_0(0,T) + \eta^*] \times [W_\delta(0,T) + u^*(t)]$$

$(i = 1,2)$ be two weak solutions to the initial-boundary value problem (3.3)–(3.5) for a given boundary influences $g \in W_0^{1,\infty}(0,T)$ and $h \in W_0^{1,\infty}(0,T)$. We set

$$\eta(t) = \eta^1(t) - \eta^2(t),\ w(t) = w^1(t) - w^2(t),\ u(t) = u^1(t) - u^2(t),\ v(t) = v^1(t) - v^2(t).$$

Since

$$b_1(w^1, v^1, \phi) - b_1(w^2, v^2, \phi) = b_1(w, v^1, \phi) + b_1(w^2, v, \phi),$$

$$b_2(v^1, v^1, \psi) - b_2(v^2, v^2, \psi) = \int_\Omega \left[v^1 v_x \psi + v v_x^2 \psi\right] dx,$$

it follows that the distributions $w(\cdot)$ and $v(\cdot)$ satisfy the following system

$$\langle \dot{w}(t), \varphi\rangle_{V_0^*;V_0} + a_1(w(t),\varphi) + b_1(w(t), v^1(t),\varphi) + b_1(w^2(t), v(t),\varphi) + b_1(w(t), u^*(t),\varphi) \\ + \frac{1}{2}\left([r_0+\eta^*]v_x(t),\varphi\right)_H = 0, \quad \forall\,\varphi\in V_0 \ \text{ and a.e. } t\in[0,T], \tag{4.1}$$

$$\langle \dot{v}(t), \psi\rangle_{V_\delta^*;V_\delta} + a_2(\dot{v}(t),\psi) + \int_\Omega \left[v^1(t)v_x(t)\psi + v(t)v_x^2(t)\psi\right]\,dx + (\mu(t)w_x(t),\psi)_H \\ + (v_x(t)u_x^*,\psi)_H = 0, \quad \forall\,\psi\in V_\delta \ \text{ and a.e. } t\in[0,T], \tag{4.2}$$

$$w(0)=0 \ \text{ in } H \text{ and } \ v(0)=0 \ \text{ in } V_\delta. \tag{4.3}$$

Due to Remark 3.1, we can choose $\varphi = w(t)$ and $\psi = v(t)$ in relations (4.1)–(4.2). Then, upon this choosing, and Lemma 2.1 (see also Remark 2.3), we have

$$b_1(w(t), v^1(t), w(t)) \overset{\text{by (3.10)}}{=} 0,$$
$$b_1(w(t), u^*(t), w(t)) \overset{\text{by (3.10)}}{=} 0,$$
$$b_1(w^2(t), v(t), w(t)) = \int_\Omega \left[\left(w^2(t)\right)_x v(t)w(t) + \frac{1}{2}w^2(t)v_x(t)w(t)\right]\,dx \\ = -\int_\Omega \left[\frac{1}{2}v_x(t)w(t) + v(t)w_x(t)\right] w^2(t)\,dx$$

and, therefore, relations (4.1)–(4.2) lead us to the equalities

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2 + \nu\|w(t)\|_{V_0}^2 - \int_\Omega \left[\frac{1}{2}v_x(t)w(t) + v(t)w_x(t)\right] w^2(t)\,dx \\ + \frac{1}{2}\int_\Omega [r_0+\eta^*]\,v_x(t)w(t)\,dx = 0, \qquad \text{for a.e. } t\in[0,T], \tag{4.4}$$

$$\frac{1}{2}\frac{d}{dt}\left[\|v(t)\|_H^2 + \int_\Omega \delta\left(vu_x(t)\right)^2\,dx\right] + \int_\Omega \left[v^1(t)v_x(t) + v(t)v_x^2(t)\right] v(t)\,dx \\ + \int_\Omega \mu(t)w_x(t)v(t)\,dx + \int_\Omega v_x(t)u_x^*(t)v(t)\,dx = 0, \qquad \text{for a.e. } t\in[0,T]. \tag{4.5}$$

For our further analysis we make use of the Young's and Gronwall's inequalities.

- (*Young's Inequality*) For all $a, b, \varepsilon > 0$ and for all $p\in(1,+\infty)$, we have

$$ab \le \frac{\varepsilon a^p}{p} + \frac{b^q}{q\varepsilon^{q/p}}, \quad \text{with} \quad q = p/(p-1); \tag{4.6}$$

- (*Gronwall's Inequality*) Let c be a positive constant. Suppose that $\varphi\in L^1(0,T)$ and $\varphi(t)$ is non-negative for a.e. $t\in[0,T]$. If $\psi\in C([0,T])$ satisfies the inequality

$$\psi(t) \le c + \int_0^t \varphi(s)\psi(s)\,ds \ \text{ for all } t\in[0,T],$$

then we have

$$\psi(t) \le c\exp\left(\int_0^t \varphi(s)\,ds\right) \ \text{ for all } t\in[0,T].$$

Taking into account these inequalities, we conclude that

$$\begin{aligned}
\frac{1}{2}\int_\Omega v_x(t)w(t)w^2(t)\,dx &\le \frac{1}{2}\|w^2(t)\|_H\|v_x(t)\|_H\|w(t)\|_{L^\infty(\Omega)} \\
&\overset{\text{by (2.4)}}{\le} 2\,\|w^2(t)\|_H\|v_x(t)\|_H\|w(t)\|_{V_0} \overset{\text{by (2.6)}}{\le} 2\,\|w^2\|_{C([0,T];H)}\|v_x(t)\|_H\|w(t)\|_{V_0} \\
&\le 2C_E\,\|w^2\|_{W_0(0,T)}\|v_x(t)\|_H\|w(t)\|_{V_0} \\
&\overset{\text{by (4.6)}}{\le} C_E\,\|w^2\|_{W_0(0,T)}\left[\varepsilon\|v_x(t)\|_H^2+\frac{1}{\varepsilon}\|w(t)\|_{V_0}^2\right]_{\left\{\varepsilon=\frac{6C_E\,\|w^2\|_{W_0(0,T)}}{\nu}\right\}} \\
&= \frac{6C_E^2\,\|w^2\|_{W_0(0,T)}^2}{\nu}\,\|v_x(t)\|_H^2+\frac{\nu}{6}\|w(t)\|_{V_0}^2 \\
&\overset{\text{by (2.1)}}{\le} \underbrace{\frac{6C_E^2\,\|w^2\|_{W_0(0,T)}^2\delta_0^{-1}}{\nu}}_{C_1}\,\|v(t)\|_{V_\delta}^2+\frac{\nu}{6}\|w(t)\|_{V_0}^2. \qquad (4.7)
\end{aligned}$$

Proceeding in the similar manner, we get

$$\begin{aligned}
\int_\Omega v(t)w_x(t)w^2(t)\,dx &\le \|w^2(t)\|_H\|w_x(t)\|_H\|v(t)\|_{L^\infty(\Omega)} \\
&\overset{\text{by (2.4)}}{\le} 2\sqrt{\delta_0^{-1}}\,\|w^2(t)\|_H\|w_x(t)\|_H\|v(t)\|_{V_\delta} \\
&\overset{\text{by (2.6)}}{\le} 2C_E\sqrt{\delta_0^{-1}}\,\|w^2\|_{W_0(0,T)}\|w_x(t)\|_H\|v(t)\|_{V_\delta} \\
&\overset{\text{by (4.6)}}{\le} C_E\sqrt{\delta_0^{-1}}\,\|w^2\|_{W_0(0,T)}\left[\varepsilon\|w_x(t)\|_H^2+\frac{1}{\varepsilon}\|v(t)\|_{V_\delta}^2\right]_{\left\{\varepsilon=\frac{\nu}{6C_E\sqrt{\delta_0^{-1}}\,\|w^2\|_{W_0(0,T)}}\right\}} \\
&= \underbrace{\frac{6C_E^2\delta_0^{-1}\,\|w^2\|_{W_0(0,T)}^2}{\nu}}_{C_2}\,\|v(t)\|_{V_\delta}^2+\frac{\nu}{6}\|w(t)\|_{V_0}^2, \qquad (4.8)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\int_\Omega [r_0+\eta^*]\,v_x(t)w(t)\,dx &\le \frac{1}{2}\|r_0\|_H\|v_x(t)\|_H\|w(t)\|_{L^\infty(\Omega)}+\frac{1}{2}\eta^*\|v_x(t)\|_H\|w(t)\|_H \\
&\overset{\text{by (2.4),(2.3)}}{\le} \left[\|r_0\|_H+\frac{\eta^*}{2}\right]\|v_x(t)\|_H\|w(t)\|_{V_0} \\
&\le \left[\|r_0\|_H+\frac{\eta^*}{2}\right]\left[\frac{\varepsilon}{2}\|v_x(t)\|_H^2+\frac{1}{2\varepsilon}\|w(t)\|_{V_0}^2\right]_{\left\{\varepsilon=\frac{3}{\nu}\left[\|r_0\|_H+\frac{\eta^*}{2}\right]\right\}} \\
&= \frac{3}{2\nu}\left[\|r_0\|_H+\frac{\eta^*}{2}\right]^2\|v_x(t)\|_H^2+\frac{\nu}{6}\|w(t)\|_{V_0}^2 \qquad (4.9) \\
&\overset{\text{by (2.1)}}{\le} \underbrace{\frac{3}{2\nu}\left[\|r_0\|_H+\frac{\eta^*}{2}\right]^2\delta_0^{-1}}_{C_3}\,\|v(t)\|_{V_\delta}^2+\frac{\nu}{6}\|w(t)\|_{V_0}^2. \qquad (4.10)
\end{aligned}$$

Combining the estimates (4.7)–(4.10), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_H^2+\frac{\nu}{2}\|w(t)\|_{V_0}^2 \le [C_1+C_2+C_3]\,\|v(t)\|_{V_\delta}^2 \quad \text{for a.e. } t\in[0,T].$$

Hence, taking into account the initial condition (4.3), after integration, we get

$$\|w(t)\|_H^2 \leq 2\,[C_1 + C_2 + C_3] \int_0^t \|v(s)\|_{V_\delta}^2\,ds \quad \text{for a.e. } t \in [0,T], \tag{4.11}$$

$$\int_0^t \|w(s)\|_{V_0}^2\,ds \leq \frac{2}{\nu}\,[C_1 + C_2 + C_3] \int_0^t \|v(s)\|_{V_\delta}^2\,ds. \tag{4.12}$$

Proceeding in the similar manner with the estimation for the equation (4.5), we obtain

$$\begin{aligned}
\int_\Omega v_x(t) u_x^*(t) v(t)\,dx &= (h(t) - g(t)) \int_\Omega v_x(t) v(t)\,dx \\
&\leq \left(\|h\|_{L^\infty(0,T)} + \|g\|_{L^\infty(0,T)}\right) \|v(t)\|_H \|v_x(t)\|_H \\
&\overset{\text{by (2.3)}}{\leq} \left(\|h\|_{L^\infty(0,T)} + \|g\|_{L^\infty(0,T)}\right) \|v_x(t)\|_H^2 \\
&\overset{\text{by (2.1)}}{\leq} \underbrace{\left(\|h\|_{L^\infty(0,T)} + \|g\|_{L^\infty(0,T)}\right) \delta_0^{-1}}_{D_1} \|v(t)\|_{V_\delta}^2.
\end{aligned} \tag{4.13}$$

Applying the similar arguments, we get

$$\begin{aligned}
&\int_\Omega \mu(t) w_x(t) v(t)\,dx \leq \|\mu(t)\|_H \|w_x(t)\|_H \|v(t)\|_{L^\infty(\Omega)} \\
&\leq 2\sqrt{\delta_0^{-1}} \|\mu(t)\|_H \|w_x(t)\|_H \|v(t)\|_{V_\delta} \leq \sqrt{\delta_0^{-1}} \|\mu(t)\|_H \left[\varepsilon \|w_x(t)\|_H^2 + \frac{1}{\varepsilon} \|v(t)\|_{V_\delta}^2\right] \\
&= \left\{\text{letting } \ \varepsilon = \sqrt{\delta_0^{-1}} \|\mu(t)\|_H\right\} = \delta_0^{-1} \|\mu(t)\|_H^2 \|w_x(t)\|_H^2 + \|v(t)\|_{V_\delta}^2 \\
&\leq \underbrace{\delta_0^{-1} \|\mu\|_{L^\infty(0,T;L^2(\Omega))}^2}_{D_0} \|w(t)\|_{V_0}^2 + \|v(t)\|_{V_\delta}^2
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&\int_\Omega v^1(t) v_x(t) v(t)\,dx \leq \|v^1(t)\|_H \|v_x(t)\|_H \|v(t)\|_{L^\infty(\Omega)} \\
&\overset{\text{by (2.5), (2.6)}}{\leq} C_A \|v^1\|_{C([0,T];H)} \|v_x(t)\|_H \|v(t)\|_H^{1/2} \|v(t)\|_{V_0}^{1/2} \\
&\overset{\text{by (2.6),(2.1)}}{\leq} C_A C_E \|v^1\|_{W_\delta(0,T)} \|v(t)\|_{V_0}^{3/2} \|v(t)\|_H^{1/2} \\
&\leq C_A C_E \|v^1\|_{W_\delta(0,T)} \|v(t)\|_{V_0}^2 \overset{\text{by (2.1)}}{\leq} \underbrace{C_A C_E \delta_0^{-1} \|v^1\|_{W_\delta(0,T)}}_{D_2} \|v(t)\|_{V_\delta}^2
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
&\int_\Omega v_x^2(t) \left[v(t)\right]^2\,dx \leq \|v_x^2(t)\|_H \|v(t)\|_H \|v(t)\|_{L^\infty(\Omega)} \\
&\overset{\text{by (2.5)}}{\leq} C_A \|v^2(t)\|_{V_0} \|v(t)\|_H^{3/2} \|v(t)\|_{V_0}^{1/2} \leq C_A \|v^2(t)\|_{V_0} \|v(t)\|_{V_0}^2 \\
&\overset{\text{by (2.1)}}{\leq} \underbrace{C_A \delta_0^{-3/2} \|v^2(t)\|_{V_\delta}}_{D_3(t)} \|v(t)\|_{V_\delta}^2.
\end{aligned} \tag{4.16}$$

In view of the estimates (4.13)–(4.16), we can conclude from (4.5) that

$$\frac{1}{2}\frac{d}{dt}\left[\|v(t)\|_H^2 + \int_\Omega \delta\left(v_x(t)\right)^2\, dx\right] \le D_0\|w(t)\|_{V_0}^2 + \left[D_1 + D_2 + D_3(t)\right]\|v(t)\|_{V_\delta}^2$$

for a.e. $t \in [0,T]$. Hence, after integration, we arrive at the fulfilment for a.e. $t \in [0,T]$ of the following inequality

$$\begin{aligned} \|v(t)\|_H^2 + \|v(t)\|_{V_\delta}^2 &\le 2D_0\int_0^t \|w(s)\|_{V_0}^2\, ds + 2\int_0^t \left[D_1 + D_2 + D_3(s)\right]\|v(s)\|_{V_\delta}^2\, ds \\ &\overset{\text{by (4.12)}}{\le} 2\int_0^t \underbrace{\left[\frac{2D_0}{\nu}\big(C_1 + C_2 + C_3\big) + \big(D_1 + D_2 + D_3(s)\big)\right]}_{C^*(s)}\|v(s)\|_{V_\delta}^2\, ds \end{aligned} \tag{4.17}$$

Gathering together the estimates (4.17) and (4.11), we finally conclude that the inequality

$$\|w(t)\|_H^2 + \|v(t)\|_{V_\delta}^2 \le 2\int_0^t \left[C_1 + C_2 + C_3 + C^*(s)\right]\left(\|w(s)\|_H^2 + \|v(s)\|_{V_\delta}^2\right)\, ds \tag{4.18}$$

holds true for a.e. $t \in [0,T]$. Taking into account the fact that

$$\int_0^T D_3(t)\, dt \le \sqrt{T}\left(\int_0^T |D_3(t)|^2\, dt\right)^{1/2} = \sqrt{T}C_A\delta_0^{-3/2}\|v^2\|_{L^2(0,T;V_\delta)} < +\infty,$$

we have $C^*(\cdot) \in L^1(0,T)$. Hence, by Gronwall's inequality, we derive from (4.18) that

$$\|w(t)\|_H^2 + \|v(t)\|_{V_\delta}^2 = 0 \quad \text{for a.e. } t \in [0,T]. \tag{4.19}$$

Now we can summarize the obtained result as follows:

Lemma 4.1. *Assume that the conditions* (3.1) *hold true. Let* $(\eta(\cdot), u(\cdot))$ *be a weak solution to the system* (3.3)–(3.5) *in the sense of Definition 3.1. Then this solution is unique in* $[W_0(0,T) + \eta^*] \times [W_\delta(0,T) + u^*]$.

5. On Existence of Weak Solutions to the Viscous Boussinesq System. In order to prove the existence of the corresponding weak solutions to the initial-boundary value problem (3.3)–(3.5), we will follow the well-known Faedo-Galerkin method which is also convenient for numerical approximations. With that in mind, we consider a finite-dimensional approximation of the problem (3.3)–(3.5). Namely, since V_0 and V_δ are separable Hilbert spaces and since $C_0^\infty(\Omega)$ and $C_0^\infty(\mathbb{R})$ are dense in V_0 and V_δ, respectively, it follows that there exist two sequences of smooth functions $\{\zeta_k\}_{k\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ such that $\{\zeta_k\}_{k\in\mathbb{N}}$ constitutes an orthogonal basis in V_0 and an orthonormal basis in H, and $\{\xi_k\}_{k\in\mathbb{N}}$ constitutes an orthonormal basis in V_δ with respect to the equivalent norm $\sqrt{\|\cdot\|_H^2 + \|\cdot\|_{V_\delta}^2}$). In particular, it means that

$$(\xi_k, \xi_n)_H = \begin{cases} \|\xi_k\|_H^2, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \qquad (\zeta_k, \zeta_n)_{V_0} = \begin{cases} \|\zeta_k\|_{V_0}^2, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases}$$
$$(\xi_k, \xi_n)_H + (\xi_k, \xi_n)_{V_\delta} = \delta_{kn}, \qquad (\zeta_k, \zeta_n)_H = \delta_{kn},$$

where δ_{kn} stands for the Kronecker delta.

Remark 5.1. *In order to construct the sequence* $\{\xi_k\}_{k\in\mathbb{N}}$ *(resp.,* $\{\zeta_k\}_{k\in\mathbb{N}}$*) with properties indicated before, we can choose as* ξ_k *(resp.,* ζ_k*) the Dirichlet eigenfunctions of the* *operator $Aw = -(\delta w')' + w$ (resp., the Dirichlet eigenfunctions of $Aw = -w''$) and normalize them with respect to the norm $\sqrt{\|\cdot\|_H^2 + \|\cdot\|_{V_\delta}^2}$ in V_δ (resp., in H).*

Following the main idea of Faedo-Galerkin method, we construct two sequences of finite-dimensional subspaces

$$V_{\delta,m} = \text{span}\,\{\xi_1, \dots, \xi_m\} \quad \text{and} \quad V_{0,m} = \text{span}\,\{\zeta_1, \dots, \zeta_m\}\,. \tag{5.1}$$

As a result, we have

$$V_{\delta,m} \subset V_{\delta,m+1}, \quad V_{0,m} \subset V_{0,m+1}, \quad \overline{\cup V_{\delta,m}} = V_\delta, \quad \overline{\cup V_{0,m}} = V_0.$$

Further, for a fixed m, we set

$$w_m(t) = \sum_{k=1}^m c_k(t)\zeta_k, \qquad v_m(t) = \sum_{k=1}^m d_k(t)\xi_k, \tag{5.2}$$

$$w_m(0) = W_m := \sum_{k=1}^m \alpha_k\zeta_k, \qquad v_m(0) = U_m := \sum_{k=1}^m \beta_k\xi_k, \tag{5.3}$$

$$\alpha_k = (\eta_0 - \eta^*, \zeta_k)_H\,, \quad \beta_k = (u_0, \xi_k)_H + ((u_0)_x, (\xi_k)_x)_{L^2(\Omega;\delta\,dx)}$$

and it is clear that

$$w_m(0) \to \eta_0 - \eta^* \text{ strongly in } H, \qquad v_m(0) \to u_0 \text{ strongly in } V_\delta. \tag{5.4}$$

DEFINITION 5.1. *We say that a couple of distributions*

$$(w_m(t) + \eta^*, v_m(t) + u^*(t)) \in [W_0(0,T) + \eta^*] \times [W_\delta(0,T) + u^*]$$

is a Galerkin approximation of the weak solution $(w(t) + \eta^*, u(t) + u^*(t))$ to the system (3.3)–(3.5) *if $w_m(t)$ and $v_m(t)$ have the representation* (5.2)–(5.3) *and satisfy the following approximate variational problem*

$$\begin{aligned} \langle \dot{w}_m(t), \zeta_k\rangle_{V_0^*;V_0} &+ a_1(w_m(t), \zeta_k) + b_1(w_m(t), v_m(t), \zeta_k) + b_1(w_m(t), u^*(t), \zeta_k) \\ &+ \frac{1}{2}\left([r_0 + \eta^*]\,[v_m(t)]_x\,, \zeta_k\right)_H + \frac{1}{2}\left([r_0 + \eta^*]\,[g(t) - h(t)]\,, \zeta_k\right)_H = 0, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \langle \dot{v}_m(t), \xi_k\rangle_{V_\delta^*;V_\delta} &+ a_2(\dot{v}_m(t), \xi_k) + b_2(v_m(t), v_m(t), \xi_k) + (\mu(t)\,[w_m(t)]_x\,, \xi_k)_H \\ &+ ([v_m(t)]_x\, u_x^*(t), \psi)_H = (f(t), \xi_k)_H - (\dot{u}^*(t), \xi_k)_H - b_2(u^*(t), u^*(t), \xi_k) \end{aligned} \tag{5.6}$$

for a.e. $t \in [0,T]$ and every $k = 1, \dots, m$.

REMARK 5.2. *Since $\dot{w}_m \in L^2(0,T;V_0)$ and $\dot{v}_m \in L^2(0,T;V_\delta)$ for any Galerkin approximation, it follows that*

$$\langle \dot{w}_m(t), \zeta_k\rangle_{V_0^*;V_0} = (\dot{w}_m(t), \zeta_k)_H \quad \textit{and} \quad \langle \dot{v}_m(t), \xi_k\rangle_{V_\delta^*;V_\delta} = (\dot{v}_m(t), \xi_k)_H\,. \tag{5.7}$$

The following assertion holds true.

PROPOSITION 5.2. *Under assumptions* (3.1)–(3.2), *for each $m \in \mathbb{N}$, there exists $T_m > 0$ such that the Galerkin approximation $(w_m(t) + \eta^*, u_m(t) + u^*(t))$ of the weak solution to the system* (3.3)–(3.5) *is unique on $[0, T_m]$. Moreover, in this case we have*

$$\left\{ \begin{array}{ll} w_m \in H^1(0,T_m;V_{0,m}), & v_m \in H^1(0,T_m;V_{\delta,m}), \\ w_m \in C([0,T_m];V_{0,m}), & v_m \in C([0,T_m];V_{\delta,m}). \end{array} \right. \tag{5.8}$$

Proof. To begin with, let us show that unknown coefficients $\{c_k(t)\}_{k\in\mathbb{N}}$ and $\{d_k(t)\}_{k\in\mathbb{N}}$ can be defined from the system (5.3),(5.5),(5.6) in a unique way on some appropriate time interval $[0,T_m]$. To do so, we note that because of orthonormality of the sequences of smooth functions $\{\zeta_k\}_{k\in\mathbb{N}}$ and $\{\xi_k\}_{k\in\mathbb{N}}$ in H, we have for all $k=1,\dots,m$

$$\begin{aligned}
(\dot w_m(t),\zeta_k)_H &= \Big(\sum_{s=1}^m \dot c_s(t)\zeta_s,\zeta_k\Big)_H = \dot c_k(t),\\
(\dot v_m(t),\xi_k)_H &= \Big(\sum_{s=1}^m \dot d_s(t)\xi_s,\xi_k\Big)_H = \dot d_k(t)\|\xi_k\|_H^2.
\end{aligned}$$

Moreover, the orthogonality of these systems in V_0 and V_δ, respectively, implies

$$\begin{aligned}
a_1(w_m(t),\zeta_k) &= \nu\,(\zeta_k',\zeta_k')_H\,c_k(t) = \nu\|\zeta_k\|_{V_0}^2 c_k(t),\\
a_2(\dot v_m(t),\xi_k) &= \sum_{s=1}^m (\delta\xi_s',\xi_k')_H\,\dot d_s(t) = \|\xi_k'\|^2_{L^2(\Omega;\delta\,dx)}\dot d_k(t),\\
b_1(w_m(t),v_m(t),\zeta_k) &= \int_\Omega \left[\sum_{s=1}^m c_s(t)\zeta_s'\right]\left[\sum_{j=1}^m d_j(t)\xi_j\right]\zeta_k\,dx\\
&\quad+\frac12\int_\Omega \left[\sum_{s=1}^m c_s(t)\zeta_s\right]\left[\sum_{j=1}^m d_j(t)\xi_j'\right]\zeta_k\,dx\\
&= \sum_{s,j=1}^m b_1(\zeta_s,\xi_j,\zeta_k)c_s(t)d_j(t),\\
b_1(w_m(t),u^*(t),\zeta_k) &= \int_\Omega \left[\sum_{s=1}^m c_s(t)\zeta_s'\right]u^*(t)\zeta_k\,dx\\
&\quad+\frac12\int_\Omega \left[\sum_{s=1}^m c_s(t)\zeta_s\right]u_x^*(t)\zeta_k\,dx\\
&= \sum_{s=1}^m b_1(\zeta_s,u^*(t),\zeta_k)c_s(t)d_j(t),
\end{aligned}$$

$$\begin{aligned}
b_2(v_m(t),v_m(t),\xi_k) &= \sum_{s,j=1}^m \left(\xi_s\xi_j',\xi_k\right)_H d_s(t)d_j(t),\\
\frac12\left(r_0\,[v_m(t)]_x+\eta^*\,[v_m(t)]_x\,,\zeta_k\right)_H &= \frac12\sum_{s=1}^m (r_0\xi_s'+\eta^*\xi_s',\zeta_k)_H\,d_s(t),\\
\left(\mu(t)\,[w_m(t)]_x\,,\xi_k\right)_H &= \sum_{s=1}^m (\mu(t)\zeta_s',\xi_k)_H\,c_s(t),\\
\left([v_m(t)]_x\,u_x^*(t),\xi_k\right)_H &= u_x^*(t)\sum_{s=1}^m (\xi_s',\xi_k)_H\,d_s(t).
\end{aligned}$$

Taking these representations into account, we set

$$
\begin{aligned}
\mathbb{C}_m(t) &= [c_1(t), \dots, c_m(t)]^t, \qquad \mathbb{C}_m^0 = [\alpha_1, \dots, \alpha_m]^t, \\
\mathbb{D}_m(t) &= [d_1(t), \dots, d_m(t)]^t, \qquad \mathbb{D}_m^0 = [\beta_1, \dots, \beta_m]^t, \\
\mathbb{F}_m(t) &= [F_1(t), \dots, F_m(t)]^t \quad \text{with} \quad F_k(t) = (f(t), \xi_k)_H - (\dot{u}^*(t), \xi_k)_H - b_2(u^*(t), u^*(t), \xi_k)), \\
\mathbb{R}_m(t) &= [R_1(t), \dots, R_m(t)]^t \quad \text{with} \quad R_k(t) = \frac{1}{2}\left([r_0 + \eta^*]\,[h(t) - g(t)], \zeta_k\right)_H, \\
\mathbb{A}_{1,m} &= \left[a_{ij}^1\right]_{i,j=1}^m \quad \text{with} \quad a_{ij}^1 = \frac{1}{2}\left(r_0 \xi_i' + \eta^* \xi_i', \zeta_j\right)_H, \\
\mathbb{A}_{2,m}(t) &= \left[a_{ij}^2(t)\right]_{i,j=1}^m \quad \text{with} \quad a_{ij}^2(t) = \left(\mu(t)\zeta_j', \xi_i\right)_H, \\
\mathbb{A}_{3,m} &= \left[a_{ij}^3\right]_{i,j=1}^m \quad \text{with} \quad a_{ij}^3 = \left(\xi_j', \xi_i\right)_H, \\
\mathbb{K}_m &= \operatorname{diag}\left\{\|\zeta_1\|_{V_0}^2, \dots, \|\zeta_m\|_{V_0}^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{C}_m^t(t)\mathbb{B}_m\mathbb{C}_m(t) &= \begin{bmatrix} \mathbb{C}_m^t(t) B_{m,1} \mathbb{C}_m(t) \\ \cdots \\ \mathbb{C}_m^t(t) B_{m,1} \mathbb{C}_m(t) \end{bmatrix}, \\
&\text{where} \quad B_{m,k} = \left[b_{ij}^k\right]_{i,j=1}^m \quad \text{with} \quad b_{ij}^k = b_1(\zeta_i, \xi_j, \zeta_k), \\
\mathbb{D}_m^t(t)\widehat{\mathbb{B}}_m\mathbb{D}_m(t) &= \begin{bmatrix} \mathbb{D}_m^t(t) \widehat{B}_{m,1} \mathbb{D}_m(t) \\ \cdots \\ \mathbb{D}_m^t(t) \widehat{B}_{m,1} \mathbb{D}_m(t) \end{bmatrix}, \\
&\text{where} \quad \widehat{B}_{m,k} = \left[\widehat{b}_{ij}^k\right]_{i,j=1}^m \quad \text{with} \quad \widehat{b}_{ij}^k = \left(\xi_i \xi_j', \xi_k\right)_H.
\end{aligned}
$$

Then the system (5.3),(5.5),(5.6) can be represented as follows

$$
\begin{cases}
\dot{\mathbb{C}}_m(t) = -\nu \mathbb{K}_{1,m}\mathbb{C}_m(t) - \mathbb{A}_{1,m}\mathbb{D}_m(t) - \mathbb{C}_m^t(t)\mathbb{B}_m\mathbb{C}_m(t) + \mathbb{R}_m(t), \\
\dot{\mathbb{D}}_m(t) = -u_x^*(t)\mathbb{A}_{3,m}\mathbb{D}_m(t) - \mathbb{A}_{2,m}(t)\mathbb{C}_m(t) - \mathbb{D}_m^t(t)\widehat{\mathbb{B}}_m\mathbb{D}_m(t) + \mathbb{F}_m(t). \\
\mathbb{C}_m(0) = \mathbb{C}_m^0, \\
\mathbb{D}_m(0) = \mathbb{D}_m^0.
\end{cases}
\tag{5.9}
$$

Since the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ is an orthogonal basis in V_δ with respect to the equivalent norm $\sqrt{\|u\|_H^2 + \|u_x\|_{L^2(\Omega;\delta\,dx)}^2}$, it follows that

$$
\operatorname{diag}\left\{\|\xi_1\|_H^2, \dots, \|\xi_m\|_H^2\right\} + \operatorname{diag}\left\{\|\xi_1'\|_{L^2(\Omega;\delta\,dx)}^2, \dots, \|\xi_m'\|_{L^2(\Omega;\delta\,dx)}^2\right\}
$$

is the identity matrix. Hence, for each $m \in \mathbb{N}$ we deal with the following Cauchy problem for a linear-quadratic system of ordinary differential equations

$$
\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \mathbb{C}_m(t) \\ \mathbb{D}_m(t) \end{bmatrix} &= \begin{bmatrix} -\nu\mathbb{K}_m & -\mathbb{A}_{1,m} \\ -\mathbb{A}_{2,m}(t) & -u_m^*(t)\mathbb{A}_{3,m} \end{bmatrix} \begin{bmatrix} \mathbb{C}_m(t) \\ \mathbb{D}_m(t) \end{bmatrix} \\
&\quad - \begin{bmatrix} \mathbb{C}_m^t(t)\mathbb{B}_m\mathbb{C}_m(t) \\ \mathbb{D}_m^t(t)\widehat{\mathbb{B}}_m\mathbb{D}_m(t) \end{bmatrix} + \begin{bmatrix} \mathbb{R}_m(t) \\ \mathbb{F}_m(t) \end{bmatrix}, \quad t > 0,
\end{aligned}
\tag{5.10}
$$

$$
\begin{bmatrix} \mathbb{C}_m(0) \\ \mathbb{D}_m(0) \end{bmatrix} = \begin{bmatrix} \mathbb{C}_m^0 \\ \mathbb{D}_m^0 \end{bmatrix}.
\tag{5.11}
$$

In view of the initial assumptions, we have

$$\mathbb{F}_m(\cdot) \in L^2(0,T;\mathbb{R}^m),\ u_x^*(\cdot) = g(\cdot) - h(\cdot) \in L^\infty(0,T),\ \mathbb{A}_{2,m}(\cdot) \in L^\infty(0,T;\mathbb{R}^{m\times m}).$$

Hence, by the Carathéodory's existence theorem, the Cauchy problem (5.10)–(5.11) admits a unique solution $[\mathbb{C}_m(t),\mathbb{D}_m(t)]^t$ in $C^1([0,T_m];\mathbb{R}^{2m})$ for an appropriate $T_m > 0$. As a result, the representation (5.2) immediately leads us to the conclusion

$$w_m \in H^1(0,T_m;V_{0,m}), \quad v_m \in H^1(0,T_m;V_{\delta,m})$$

and, therefore, $w_m(\cdot) \in W_0(0,T_m)$ and $v_m(\cdot) \in W_\delta(0,T_m)$. It remains to note that the rest functional properties that were indicated in (5.8), are the direct consequence of the Sobolev embedding theorem (see Remark 2.3). □

Remark 5.3. *Since the nonlinearity in the right-hand side of the system* (5.9) *is locally Lipschitz continuous with respect to the vector-function* $[\mathbb{C}_m(t),\mathbb{D}_m(t)]^t$*, by the well-known results of ODEs theory (see [19]) it follows that the unique solution to the Cauchy problem* (5.10)–(5.11) *can be extended by continuity from* $[0,T_m]$ *to any larger interval. Hence, we can suppose that the intervals* $[0,T_m]$ *can be chosen such that (*$T_m \ge T$ *for* $m = 1,2,\dots$*).*

Our next intention is to show that the sequence $\{(w_m(t),v_m(t))\}_{m\in\mathbb{N}}$ of Galerkin approximations possesses some compactness properties in an appropriate topology. To do so, we begin with the following technical result.

Lemma 5.3. *Under assumptions* (3.1)–(3.2)*, there exists a constant* $C^* > 0$*, independent of* $m \in \mathbb{N}$*, such that* $C^* = C^*\left(\|h\|_{W_0^{1,\infty}(0,T)}, \|g\|_{W_0^{1,\infty}(0,T)}\right)$ *and*

$$\|w_m(t)\|_H^2 + \|v_m(t)\|_H^2 + \|v_m(t)\|_{V_\delta}^2 \le C^* \quad \text{for all } m \in \mathbb{N} \text{ and } t \in [0,T]. \tag{5.12}$$

In particular, the following estimates

$$\|v_m(t)\|_{V_\delta}^2 \le \Phi^{-1}\left(T + \Phi\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2\right)\right) \quad \forall\, t \in [0,T], \tag{5.13}$$

$$\begin{aligned}\|w_m(t)\|_H^2 &\le \|\eta_0 - \eta^*\|_H^2 + C_1^* T\\ &\quad + C_0^*\, T\, \Phi^{-1}\left(T + \Phi\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2\right)\right), \quad \forall\, t \in [0,T]\end{aligned} \tag{5.14}$$

hold true, where

$$\begin{cases} C_0^* = \dfrac{1}{\nu\delta_0}\left[\|r_0\|_H + \eta^*\right]^2, \\ C_1^* = \dfrac{1}{\nu}\left[\|r_0\|_H + \eta^*\right]^2\left(\|g\|_{L^\infty(0,T)} + \|h\|_{L^\infty(0,T)}\right)^2, \\ C_2^* = \sqrt{\delta_0^{-1}}\left[\|f\|_{L^\infty(0,T;H)} + \dfrac{3}{2}\widehat{C} + \widehat{C}^2\right], \\ C_3^* = \delta_0^{-1}\left(\widehat{C} + \dfrac{1}{2\nu}\|\mu\|_{L^\infty(0,T;H)}^2\right), \\ \widehat{C} = \|g\|_{W_0^{1,\infty}(0,T)} + \|h\|_{W_0^{1,\infty}(0,T)}, \end{cases} \tag{5.15}$$

and

$$\Phi(q) := \int_\varepsilon^q \frac{ds}{C_1^* + C_2^*\sqrt{s} + (C_0^* + C_3^*)\, s + \delta_0^{-3/2}\, s\sqrt{s}}, \tag{5.16}$$

for some positive value $\varepsilon > 0$.

Proof. Multiplying the equations (5.5)-(5.6) by $c_k(t)$ and $d_k(t)$, respectively, and summing both equalities for $k = 1, \dots, m$, we get (see also (5.7))

$$
\begin{aligned}
&(\dot{w}_m(t), w_m(t))_H + a_1(w_m(t), w_m(t)) + \frac{1}{2}\left([r_0+\eta^*]\,[v_m(t)]_x\,, w_m(t)\right)_H \\
&\qquad + \frac{1}{2}\left([r_0+\eta^*]\,[g(t)-h(t)]\,, w_m(t)\right)_H = 0, \quad \text{for a.e. } t\in[0,T], \qquad (5.17)\\
&(\dot{v}_m(t), v_m(t))_H + a_2(\dot{v}_m(t), v_m(t)) + b_2(v_m(t), v_m(t), v_m(t)) + \left(\mu(t)\,[w_m(t)]_x\,, v_m(t)\right)_H \\
&\qquad + \left([v_m(t)]_x\, u^*_x(t), v_m(t)\right)_H = (f(t), v_m(t))_H - (\dot{u}^*(t), v_m(t))_H \\
&\qquad - b_2(u^*(t), u^*(t), v_m(t)), \ \text{for a.e. } t\in[0,T]. \qquad (5.18)
\end{aligned}
$$

We note that by (5.8) and Lemma 2.1, for a.e. $t\in[0,T]$, we have

$$
\begin{cases}
(\dot{w}_m(t), w_m(t))_H = \dfrac{1}{2}\dfrac{d}{dt}\|w_m(t)\|^2_H, \\
(\dot{v}_m(t), v_m(t))_H + a_2(\dot{v}_m(t), v_m(t)) = \dfrac{1}{2}\dfrac{d}{dt}\left[\|v_m(t)\|^2_H + \|v_m(t)\|^2_{V_\delta}\right], \\
a_1(w_m(t), w_m(t)) = \nu\|w_m(t)\|^2_{V_0}.
\end{cases} \qquad (5.19)
$$

Moreover, the Holder's, Young's, and Agmon's inequalities imply the following estimate for the rest terms in (5.17)–(5.18):

$$
\begin{aligned}
&\frac{1}{2}\left(r_0\,[v_m(t)]_x + \eta^*\,[v_m(t)]_x\,, w_m(t)\right)_H \le \frac{1}{2}\|r_0+\eta^*\|_H\|[v_m(t)]_x\|_H\|w_m(t)\|_{L^\infty(\Omega)} \\
&\quad \overset{\text{by (2.1), (2.4)}}{\le} \left[\|r_0\|_H + \eta^*\right]\sqrt{\delta_0^{-1}}\|v_m(t)\|_{V_\delta}\|w_m(t)\|_{V_0} \\
&\quad \overset{\text{by (4.6)}}{\le} \left[\|r_0\|_H + \eta^*\right]\sqrt{\delta_0^{-1}}\left[\frac{\varepsilon}{2}\|v_m(t)\|^2_{V_\delta} + \frac{1}{2\varepsilon}\|w_m(t)\|^2_{V_0}\right]_{\left\{\varepsilon=\frac{2}{\nu}\left[\|r_0\|_H+\eta^*\right]\sqrt{\delta_0^{-1}}\right\}} \\
&\quad = \underbrace{\frac{1}{\nu\delta_0}\left[\|r_0\|_H + \eta^*\right]^2}_{C_0^*}\|v_m(t)\|^2_{V_\delta} + \frac{\nu}{4}\|w_m(t)\|^2_{V_0}; \qquad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}\left([r_0+\eta^*]\,[g(t)-h(t)]\,, w_m(t)\right)_H \le \frac{1}{2}\|r_0+\eta^*\|_H\left(\|g\|_{L^\infty(0,T)} + \|h\|_{L^\infty(0,T)}\right)\|w_m(t)\|_H \\
&\le \left[\|r_0\|_H + \eta^*\right]\left(\|g\|_{L^\infty(0,T)} + \|h\|_{L^\infty(0,T)}\right)\|w_m(t)\|_{V_0} \\
&\overset{\text{by (4.6)}}{\le} \left(\|g\|_{L^\infty(0,T)} + \|h\|_{L^\infty(0,T)}\right)\left[\frac{\varepsilon}{2}\left[\|r_0\|_H + \eta^*\right]^2 + \frac{1}{2\varepsilon}\|w_m(t)\|^2_{V_0}\right] \\
&\qquad \left\{\varepsilon = \frac{2}{\nu}\left(\|g\|_{L^\infty(0,T)} + \|h\|_{L^\infty(0,T)}\right)\right\} \\
&= \underbrace{\frac{1}{\nu}\left[\|r_0\|_H + \eta^*\right]^2\left(\|g\|_{L^\infty(0,T)} + \|h\|_{L^\infty(0,T)}\right)^2}_{C_1^*} + \frac{\nu}{4}\|w_m(t)\|^2_{V_0}; \qquad (5.21)
\end{aligned}
$$

$$
(f(t), v_m(t))_H \le \|f(t)\|_H \|v_m(t)\|_H \le \sqrt{\delta_0^{-1}} \|f\|_{L^\infty(0,T;H)} \|v_m(t)\|_{V_\delta}; \tag{5.22}
$$

$$
\begin{aligned}
(\dot{u}^*(t), v_m(t))_H &\le \|\dot{u}^*(t)\|_H \|v_m(t)\|_H \le \frac{3}{2}\sqrt{\delta_0^{-1}} \left(|\dot{g}(t)| + |\dot{h}(t)|\right) \|v_m(t)\|_{V_\delta} \\
&\le \frac{3}{2}\sqrt{\delta_0^{-1}} \underbrace{\left(\|g\|_{W_0^{1,\infty}(0,T)} + \|h\|_{W_0^{1,\infty}(0,T)}\right)}_{\widehat{C}} \|v_m(t)\|_{V_\delta};
\end{aligned} \tag{5.23}
$$

$$
\begin{aligned}
b_2(u^*(t), u^*(t), v_m(t)) &= \int_\Omega u^*(t) u^*_x(t) v_m(t)\, dx \le (|g(t)| + |h(t)|) \|u^*_x(t)\|_H \|v_m(t)\|_H \\
&\le \left(\|g\|_{W_0^{1,\infty}(0,T)} + \|h\|_{W_0^{1,\infty}(0,T)}\right)^2 \sqrt{\delta_0^{-1}} \|v_m(t)\|_{V_\delta} \\
&= \widehat{C}^2 \sqrt{\delta_0^{-1}} \|v_m(t)\|_{V_\delta};
\end{aligned} \tag{5.24}
$$

$$
\begin{aligned}
([v_m(t)]_x\, u^*_x(t), v_m(t))_H &\le (|g(t)| + |h(t)|) \|v_m(t)\|^2_{V_0} \le (|g(t)| + |h(t)|)\, \delta_0^{-1} \|v_m(t)\|^2_{V_\delta} \\
&\le \delta_0^{-1} \widehat{C} \|v_m(t)\|^2_{V_\delta};
\end{aligned} \tag{5.25}
$$

$$
\begin{aligned}
b_2(v_m(t), v_m(t), v_m(t)) &\le \int_\Omega v_m^2(t) |[v_m(t)]_x|\, dx \le \|v_m(t)\|_H \|[v_m(t)]_x\|_H \|v_m(t)\|_{L^\infty(\Omega)} \\
&\le \sqrt{\delta_0^{-1}}\ \|v_m(t)\|_H \|[v_m(t)]_x\|_H \|v_m(t)\|_{V_\delta} \\
&\le \delta_0^{-1}\ \|v_m(t)\|_H \|v_m(t)\|^2_{V_\delta} \le \delta_0^{-3/2}\ \|v_m(t)\|^3_{V_\delta};
\end{aligned} \tag{5.26}
$$

$$
\begin{aligned}
&\left(\mu(t)\, [w_m(t)]_x\,, v_m(t)\right)_H \le \|\mu(t)\|_H \|[w_m(t)]_x\|_H \|v_m(t)\|_{L^\infty(\Omega)} \\
&\quad\le \sqrt{\delta_0^{-1}}\ \|\mu\|_{L^\infty(0,T;H)} \|w_m(t)\|_{V_0} \|v_m(t)\|_{V_\delta} \\
&\quad\le \sqrt{\delta_0^{-1}}\ \|\mu\|_{L^\infty(0,T;H)} \left[\frac{\varepsilon}{2}\|w_m(t)\|^2_{V_0} + \frac{1}{2\varepsilon}\|v_m(t)\|^2_{V_\delta}\right]_{\left\{\varepsilon = \frac{\nu}{\sqrt{\delta_0^{-1}}\ \|\mu\|_{L^\infty(0,T;H)}}\right\}} \\
&\quad\le \frac{\nu}{2}\|w_m(t)\|^2_{V_0} + \frac{1}{2\nu}\delta_0^{-1}\ \|\mu\|^2_{L^\infty(0,T;H)} \|v_m(t)\|^2_{V_\delta}.
\end{aligned} \tag{5.27}
$$

Combining the estimates (5.20)–(5.27) together with the representations (5.19), we derive from (5.17)–(5.18) the following inequalities

$$
\frac{1}{2}\frac{d}{dt}\|w_m(t)\|^2_H + \frac{\nu}{2}\|w_m(t)\|^2_{V_0} \le C_0^* \|v_m(t)\|^2_{V_\delta} + C_1^*, \tag{5.28}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|v_m(t)\|^2_H + \|v_m(t)\|^2_{V_\delta}\right] \le \underbrace{\sqrt{\delta_0^{-1}}\left[\|f\|_{L^\infty(0,T;H)} + \frac{3}{2}\widehat{C} + \widehat{C}^2\right]}_{C_2^*} \|v_m(t)\|_{V_\delta} \\
&\quad + \underbrace{\delta_0^{-1}\left(\widehat{C} + \frac{1}{2\nu}\|\mu\|^2_{L^\infty(0,T;H)}\right)}_{C_3^*} \|v_m(t)\|^2_{V_\delta} + \delta_0^{-3/2}\ \|v_m(t)\|^3_{V_\delta} + \frac{\nu}{2}\|w_m(t)\|^2_{V_0}.
\end{aligned} \tag{5.29}
$$

Then it follows from (5.28)–(5.29) that

$$
\frac{d}{dt}\left[\|w_m(t)\|^2_H + \|v_m(t)\|^2_H + \|v_m(t)\|^2_{V_\delta}\right] \le \Psi\left(\|v_m(t)\|_{V_\delta}\right), \tag{5.30}
$$

where

$$\Psi(z) := C_1^* + C_2^* z + (C_0^* + C_3^*)\, z^2 + \delta_0^{-3/2}\, z^3.$$

Thus, we arrive at the differential inequalities

$$\frac{d}{dt}\left[\|w_m(t)\|_H^2 + \|v_m(t)\|_H^2 + \|v_m(t)\|_{V_\delta}^2\right] \le \Psi\left(\|v_m(t)\|_{V_\delta}\right), \tag{5.31}$$

$$\frac{d}{dt}\|w_m(t)\|_H^2 \le C_0^*\|v_m(t)\|_{V_\delta}^2 + C_1^*, \tag{5.32}$$

$$\|w_m(0)\|_H^2 + \|v_m(0)\|_H^2 + \|v_m(0)\|_{V_\delta}^2 \overset{\text{by (5.3)}}{=} \sum_{k=1}^m \left(\alpha_k^2 + \beta_k^2\right). \tag{5.33}$$

By Parseval's identity, we have the following monotonicity property

$$\sum_{k=1}^m \left(\alpha_k^2 + \beta_k^2\right) \nearrow \sum_{k=1}^\infty \left(\alpha_k^2 + \beta_k^2\right) = \|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2 \quad \text{as } m \to \infty.$$

Hence, the sequence $\left\{\|w_m(0)\|_H^2 + \|v_m(0)\|_H^2 + \|u_m(0)\|_{V_\delta}^2\right\}_{m\in\mathbb{N}}$ is bounded. Therefore, passing to the integral form of (5.31)–(5.33), we obtain

$$\|v_m(t)\|_{V_\delta}^2 \le \|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2 + \int_0^t \widehat{\Psi}\left(\|v_m(s)\|_{V_\delta}^2\right) ds, \quad t \in [0,T], \tag{5.34}$$

$$\|w_m(t)\|_H^2 \le \|\eta_0 - \eta^*\|_H^2 + C_1^* T + C_0^* \int_0^t \|v_m(s)\|_{V_\delta}^2\, ds, \quad t \in [0,T] \tag{5.35}$$

with

$$\widehat{\Psi}\,(z) := C_1^* + C_2^*\sqrt{z} + (C_0^* + C_3^*)\, z + \delta_0^{-3/2}\, z\sqrt{z}. \tag{5.36}$$

Putting

$$y(t) := \int_0^t \widehat{\Psi}\left(\|v_m(s)\|_{V_\delta}^2\right) ds, \quad t \in [0,T],$$

we have $y(0) = 0$, and the relation (5.34) leads us to the inequality

$$\dot{y}(t) \le \widehat{\Psi}\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2 + y(t)\right), \quad t \in [0,T].$$

Then, by integration on $[0,t]$, we get

$$\int_0^{y(t)} \frac{ds}{\widehat{\Psi}\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2 + s\right)} \le t, \quad t \in [0,T].$$

Setting $\Phi(q) := \int_\varepsilon^q \frac{ds}{\widehat{\Psi}(s)}$, where $\varepsilon > 0$ is some small enough positive value, the previous inequality can be rewritten as follows

$$\Phi\left(y(t) + \|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2\right) \le t + \Phi\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2\right),$$

that is (see (5.34) and (5.36))

$$\|v_m(t)\|_{V_\delta}^2 \le \Phi^{-1}\left(t + \Phi\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2\right)\right) \quad \forall\, t \in [0,T]. \tag{5.37}$$

It is worth to notice that the function $[0,+\infty) \ni t \mapsto \Psi(t) \in [0,+\infty)$ is monotonically increasing. Hence, there exists a unique inverse function $q \mapsto \Phi^{-1}(q)$ with the same property. As a result, (5.37) and (5.35) immediately imply the required estimates (5.13)–(5.14). □

As an obvious consequence of Lemma 5.3 and inequality

$$\begin{aligned} \nu \int_0^T \|w_m(t)\|_{V_0}^2\,dt &\le \|w_m(0)\|_H^2 + C_1^* T + C_0^* \int_0^T \|v_m(t)\|_{V_\delta}^2\,dt \\ &\overset{\text{by (5.13)}}{\le} \|w_m(0)\|_H^2 + C_1^* T + C_0^*\, T\, \Phi^{-1}\left(T + \Phi\left(\|\eta_0 - \eta^*\|_H^2 + \|u_0\|_H^2 + \|u_0\|_{V_\delta}^2\right)\right), \end{aligned} \tag{5.38}$$

coming from (5.28), we have the following result.

Corollary 5.4. *The sequence of Galerkin approximations* $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$ *is such that*

$$\{w_m(t)\}_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0,T;H), \tag{5.39}$$

$$\{w_m(t)\}_{m\in\mathbb{N}} \quad \text{is bounded in } L^2(0,T;V_0), \tag{5.40}$$

$$\{v_m(t)\}_{m\in\mathbb{N}} \quad \text{is bounded in } L^\infty(0,T;V_\delta), \tag{5.41}$$

uniformly with respect to m.

We now proceed with an estimate of the norms of $\{(\dot w_m(t), \dot v_m(t))\}_{m\in\mathbb{N}}$ in appropriate spaces.

Lemma 5.5. *There exist constants $\widehat{C}_k$ $(k = 1,\dots,3)$ and $\widehat{D}_i$ $(i = 1,\dots,5)$ independent of m such that the following estimates hold:*

$$\|\dot w_m(t)\|_{V_0^*} \le \widehat{C}_3 + \sqrt{2}\max\left\{\nu + 2\widehat{C}, \widehat{C}_2\right\}\sqrt{\Phi_m(t)} + \frac{\widehat{C}_1}{2}\Phi_m(t), \tag{5.42}$$

$$\|\dot v_m(t)\|_H + \|\dot v_m(t)\|_{V_\delta} \le \widehat{D}_3 + \widehat{D}_4 + \sqrt{2}\left(\widehat{D}_1 + \widehat{D}_2\right)\sqrt{\Phi_m(t)} + \widehat{D}_5\Phi(t) \tag{5.43}$$

for all $m \in \mathbb{N}$ and a.e. $t \in [0,T]$, where $\widehat{C}$ is defined in (5.23) *and*

$$\Phi_m(t) := \|w_m(t)\|_H^2 + \|v_m(t)\|_H^2 + \|v_m(t)\|_{V_\delta}^2 \overset{by\ (5.12)}{\le} C^*. \tag{5.44}$$

Proof. Let $z \in V_0$ and $v \in V_\delta$ be arbitrary elements, and let $m \in \mathbb{N}$ be a fixed positive integer. Then we have a decomposition

$$z = w + w^0, \quad v = u + u^0,$$

where $w \in V_{0,m}$, $w^0 \in V_{0,m}^\perp$, $u \in V_{\delta,m}$, and $u^0 \in V_{\delta,m}^\perp$. Hence,

$$\|w\|_{V_0} \le \|z\|_{V_0} \quad \text{and} \quad \|u\|_{V_\delta} \le \|v\|_{V_\delta}. \tag{5.45}$$

Since (see Remark 5.2)

$$\langle \dot w_m(t), z\rangle_{V_0^*;V_0} = (\dot w_m(t), w)_H \quad \text{and} \quad \langle \dot v_m(t), v\rangle_{V_\delta^*;V_\delta} = (\dot v_m(t), u)_H, \tag{5.46}$$

it follows from (5.5)–(5.6) that the following equalities

$$\begin{aligned} (\dot w_m(t), z)_H &= -a_1(w_m(t), w) - b_1(w_m(t), v_m(t), w) - b_1(w_m(t), u^*(t), w) \\ &\quad - \frac{1}{2}\left([r_0 + \eta^*]\,[v_m(t)]_x, w\right)_H - \frac{1}{2}\left([r_0 + \eta^*]\,[g(t) - h(t)], w\right)_H, \end{aligned} \tag{5.47}$$

$$\begin{aligned} (\dot v_m(t), v)_H + a_2(\dot v_m(t), v) &= -b_2(v_m(t), v_m(t), u) - (\mu(t)\,[w_m(t)]_x, u)_H \\ &\quad - ([v_m(t)]_x\, u_x^*(t), u)_H + (f(t), u)_H - (\dot u^*(t), u)_H - b_2(u^*(t), u^*(t), u) \end{aligned} \tag{5.48}$$

hold true for a.e. $t \in [0,T]$.

Then, by analogy with (4.7)–(4.10), we get

$$
\begin{aligned}
&|a_1(w_m(t),w)| \le \nu \|w_m(t)\|_{V_0}\|w\|_{V_0},\\
&|b_1(w_m(t),v_m(t),w)| \le \left(\frac{1}{2}\|[v_m(t)]_x\|_H\|w_m(t)\|_{L^\infty(\Omega)} + \|[w_m(t)]_x\|_H\|v_m(t)\|_{L^\infty(\Omega)}\right)\|w\|_H\\
&\qquad \overset{\text{by } (2.1),(2.4),(2.3)}{\le} \underbrace{3\sqrt{\delta_0^{-1}}}_{\widehat{C}_1}\|v_m(t)\|_{V_\delta}\|w_m(t)\|_{V_0}\|w\|_{V_0},\\
&|b_1(w_m(t),u^*(t),w)| \le \left(\frac{1}{2}\|[u^*(t)]_x\|_H\|w_m(t)\|_{L^\infty(\Omega)} + \|[w_m(t)]_x\|_H\|u^*(t)\|_{L^\infty(\Omega)}\right)\|w\|_H\\
&\qquad \overset{\text{by } (3.12)}{\le} \underbrace{\left(\|g\|_{W_0^{1,\infty}(0,T)} + \|h\|_{W_0^{1,\infty}(0,T)}\right)}_{\widehat{C}}\left(\|w_m(t)\|_{V_0} + \|[w_m(t)]_x\|_H\right)\|w\|_{V_0}\\
&\qquad = 2\widehat{C}\|w_m(t)\|_{V_0}\|w\|_{V_0},\\
&\left|\frac{1}{2}\left(r_0\,[u_m(t)]_x + \eta^*\,[u_m(t)]_x\,,w\right)_H\right| \le \frac{1}{2}\|r_0\|_H\|[v_m(t)]_x\|_H\|w\|_{L^\infty(\Omega)} + \frac{1}{2}\eta^*\|[v_m(t)]_x\|_H\|w\|_H\\
&\qquad \le \underbrace{\sqrt{\delta_0^{-1}}\left(\|r_0\|_H + \frac{1}{2}\eta^*\right)}_{\widehat{C}_2}\|v_m(t)\|_{V_\delta}\|w\|_{V_0},\\
&\left|\frac{1}{2}\left([r_0+\eta^*]\,[g(t)-h(t)]\,,w\right)_H\right| \le \frac{1}{2}\|r_0\|_H\|g(t)-h(t)\|_H\|w\|_{L^\infty(\Omega)} + \frac{1}{2}\eta^*\|g(t)-h(t)\|_H\|w\|_H\\
&\qquad \le \underbrace{\widehat{C}\left(\|r_0\|_H + \frac{1}{2}\eta^*\right)}_{\widehat{C}_3}\|w\|_{V_0}.
\end{aligned}
$$

Hence, combining this estimates with the representation (5.47), we obtain

$$
\begin{aligned}
|(\dot{w}_m(t),z)_H| &\le \left[\widehat{C}_3 + \left(\nu + 2\widehat{C}\right)\|w_m(t)\|_{V_0} + \widehat{C}_1\|v_m(t)\|_{V_\delta}\|w_m(t)\|_{V_0} + \widehat{C}_2\|v_m(t)\|_{V_\delta}\right]\|w\|_{V_0}\\
&\le \left[\widehat{C}_3 + \left(\nu + 2\widehat{C}\right)\|w_m(t)\|_{V_0} + \widehat{C}_1\|v_m(t)\|_{V_\delta}\|w_m(t)\|_{V_0} + \widehat{C}_2\|v_m(t)\|_{V_\delta}\right]\|z\|_{V_0}.
\end{aligned}
$$

Then, by definition of the norm in V_0^*, we may write

$$
\begin{aligned}
\|\dot{w}_m(t)\|_{V_0^*} &\le \widehat{C}_3 + \left(\nu + 2\widehat{C}\right)\|w_m(t)\|_{V_0} + \widehat{C}_1\|v_m(t)\|_{V_\delta}\|w_m(t)\|_{V_0} + \widehat{C}_2\|v_m(t)\|_{V_\delta}\\
&\le \widehat{C}_3 + \sqrt{2}\max\left\{\nu + 2\widehat{C}, \widehat{C}_2\right\}\sqrt{\Phi_m(t)} + \frac{\widehat{C}_1}{2}\Phi_m(t)\\
&\overset{\text{by } (5.12)}{\le} \widetilde{C}, \quad \text{for all } m \in \mathbb{N} \text{ and a.e. } t \in [0,T],
\end{aligned} \tag{5.49}
$$

where the constant $\widetilde{C}$ does not depend on m.

Now we can proceed with the similar estimation for the equation (5.48). Using the

Schwarz, Friedrich's, and Young'ss inequalities, we infer

$$\begin{aligned}\left|\left(\mu(t)\left[w_m(t)\right]_x, u\right)_H\right| &\le \|\mu\|_{L^\infty(0,T;H)}\|\left[w_m(t)\right]_x\|_H\|u\|_{L^\infty(\Omega)}\\ &\le \underbrace{2\sqrt{\delta_0^{-1}}\|\mu\|_{L^\infty(0,T;H)}}_{\widehat{D}_1}\|w_m(t)\|_{V_0}\|u\|_{V_\delta},\\ \left|\left(\left[v_m(t)\right]_x u_x^*(t), u\right)_H\right| &\le \widehat{C}\|\left[v_m(t)\right]_x\|_H\|u\|_H \le \underbrace{\delta_0^{-1}\widehat{C}}_{\widehat{D}_2}\|v_m(t)\|_{V_\delta}\|u\|_{V_\delta},\\ \left|\left(f(t), u\right)_H - \left(\dot{u}^*(t), u\right)_H\right| &\le \|f\|_{L^\infty(0,T;H)}\|u\|_H + \|\dot{u}^*(t)\|_H\|u\|_H\\ &\le \underbrace{\sqrt{\delta_0^{-1}}\Big(\|f\|_{L^\infty(0,T;H)} + \|\dot{h}\|_{L^\infty(0,T)} + \|\dot{g}\|_{L^\infty(0,T)}\Big)}_{\widehat{D}_3}\|u\|_{V_\delta},\end{aligned}$$

and

$$\begin{aligned}|b_2(v_m(t), v_m(t), u)| &= \left|\int_\Omega u[v_m(t)]_x v_m(t)\,dx\right| \le \|[v_m(t)]_x\|_H\|u\|_H\|v_m(t)\|_{L^\infty(\Omega)}\\ &\le \underbrace{2\delta_0^{-3/2}}_{\widehat{D}_5}\|v_m(t)\|_{V_\delta}^2\|u\|_{V_\delta},\\ |b_2(u^*(t), u^*(t), u)| &= \left|\int_\Omega u u_x^*(t) u^*(t)\,dx\right| \le \|u_x^*(t)\|_H\|u\|_H\|u^*(t)\|_{L^\infty(\Omega)}\\ &\le \underbrace{\sqrt{\delta_0^{-1}}\widehat{C}^2}_{\widehat{D}_4}\|u\|_{V_\delta}.\end{aligned}$$

In view of these estimates, we deduce from (5.48)

$$\begin{aligned}(\dot{v}_m(t), v)_H + a_2(\dot{v}_m(t), v) &= (\dot{v}_m(t), v)_H + (\dot{v}_m(t), v)_{V_\delta}\\ &\le \left[\widehat{D}_1\|w_m(t)\|_{V_0} + \widehat{D}_2\|v_m(t)\|_{V_\delta} + \widehat{D}_3 + \widehat{D}_4 + \widehat{D}_5\|v_m(t)\|_{V_\delta}^2\right]\|u\|_{V_\delta}\\ &\le \left[\widehat{D}_1\|w_m(t)\|_{V_0} + \widehat{D}_2\|v_m(t)\|_{V_\delta} + \widehat{D}_3 + \widehat{D}_4 + \widehat{D}_5\|v_m(t)\|_{V_\delta}^2\right]\|v\|_{V_\delta}.\end{aligned}$$

Hence,

$$\begin{aligned}\|\dot{v}_m(t)\|_H + \|\dot{v}_m(t)\|_{V_\delta} &\le \widehat{D}_1\|w_m(t)\|_{V_0} + \widehat{D}_2\|v_m(t)\|_{V_\delta} + \widehat{D}_3 + \widehat{D}_4 + \widehat{D}_5\|v_m(t)\|_{V_\delta}^2\\ &\le \widehat{D}_3 + \widehat{D}_4 + \sqrt{2}\left(\widehat{D}_1 + \widehat{D}_2\right)\sqrt{\Phi_m(t)} + \widehat{D}_5\Phi(t)\\ &\overset{\text{by (5.12)}}{\le} \widetilde{D}, \quad \text{for all } m \in \mathbb{N} \text{ and a.e. } t \in [0,T],\end{aligned} \tag{5.50}$$

where the constant $\widetilde{D}$ does not depend on m. The proof is complete. □

Remark 5.4. *Lemma 5.5 shows that the sequence $\{\dot{w}_m(t)\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V_0^*)$ while the sequence $\{\dot{v}_m(t)\}_{m\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V_\delta)$. Combining this fact with Corollary 5.4, we conclude that*

$$\{w_m(t)\}_{m\in\mathbb{N}} \quad \textit{is bounded in } L^\infty(0,T;H)\cap W_0(0,T), \tag{5.51}$$

$$\{\dot{w}_m(t)\}_{m\in\mathbb{N}} \quad \textit{is bounded in } L^\infty(0,T;V_0^*), \tag{5.52}$$

$$\{v_m(t)\}_{m\in\mathbb{N}} \quad \textit{is bounded in } W^{1,\infty}(0,T;V_\delta). \tag{5.53}$$

Hence, by Banach-Alaoglu compactness theorem, we can deduce that there exists a subsequence of the sequence of Galerkin approximations $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$, still denoted by the same suffix m, such that, as $m \to \infty$ (see Lions [16, Theorem 5.1] and Simon [20]),

$$w_m \rightharpoonup w \text{ weakly in } L^2(0,T;V_0), \tag{5.54}$$

$$w_m \overset{*}{\rightharpoonup} w \text{ weakly-}* \text{ in } L^\infty(0,T;H), \tag{5.55}$$

$$\dot{w}_m \rightharpoonup z \text{ weakly in } L^2(0,T;V_0^*) \text{ and weakly-}* \text{ in } L^\infty(0,T;V_0^*), \tag{5.56}$$

$$v_m \to v \text{ strongly in } L^p(0,T;V_\delta) \text{ for any } p \in (1,\infty) \text{ by (5.53)}, \tag{5.57}$$

$$v_m \overset{*}{\rightharpoonup} v \text{ weakly-}* \text{ in } L^\infty(0,T;V_\delta), \tag{5.58}$$

$$\dot{v}_m \rightharpoonup u \text{ weakly in } L^2(0,T;V_\delta) \text{ and weakly-}* \text{ in } L^\infty(0,T;V_\delta), \tag{5.59}$$

where $z = \dot{w}$ in the sense of $\mathcal{D}'(0,T;V_0^*)$, and $u = \dot{v}$ as elements of $\mathcal{D}'(0,T;V_\delta)$. Indeed, in view of the definition of generalized derivative, we have

$$\int_0^T \langle \dot{w}_m(t), \zeta\rangle_{V_0^*;V_0}\, \varphi(t)\, dt = -\int_0^T (w_m(t),\zeta)_{V_0} \frac{\partial\varphi}{\partial t}\, dt,\ \forall\, \zeta \in V_0,\ \forall\, \varphi \in C_0^\infty(0,T).$$

Then (5.54) *implies that $\dot{w}_m \to \dot{w}$ in the sense of distributions $\mathcal{D}'(0,T;V_0^*)$. Since, the limit in $\mathcal{D}'(0,T;V_0^*)$ is unique, in follows that $z = \dot{w}$. The similar arguments show that $u = \dot{v}$ in $\mathcal{D}'(0,T;V_\delta)$.*

In order to proceed further, we need a couple of the following technical results.

Proposition 5.6. *Let $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$ be a sequence with properties* (5.51)–(5.53) *and* (5.54)–(5.59), *and let $u \in L^2(0,T;V_0)$ be an arbitrary distribution. Then*

$$\lim_{m\to\infty} \int_0^T b_1(w_m(t), v_m(t), u(t))\, dt = \int_0^T b_1(w(t), v(t), u(t))\, dt. \tag{5.60}$$

Proof. Following the definition of the trilinear form b_1 (see (3.8)), we have

$$\begin{aligned}
\int_0^T b_1(w_m(t), v_m(t), u(t))\, dt &= \int_0^T \int_\Omega \left[[w_m(t)]_x v_m(t) u(t) + \frac{1}{2} w_m(t) [v_m(t)]_x u(t) \right] dxdt \\
&= \int_0^T \int_\Omega \left([w_m(t)]_x - [w(t)]_x\right) v(t)u(t)\, dxdt + \int_0^T \int_\Omega [w_m(t)]_x \left(v_m(t) - v(t)\right) u(t)\, dxdt \\
&\quad + \int_0^T \int_\Omega [w(t)]_x v(t) u(t)\, dxdt + \frac{1}{2}\int_0^T \int_\Omega \left(w_m(t) - w(t)\right) [v(t)]_x u(t)\, dxdt \\
&\quad + \frac{1}{2}\int_0^T \int_\Omega w_m(t) \left([v_m(t)]_x - [v(t)]_x\right) u(t)\, dxdt + \frac{1}{2}\int_0^T \int_\Omega w(t)[v(t)]_x u(t)\, dxdt \\
&= J_1^m + J_2^m + \int_0^T \int_\Omega [w(t)]_x v(t) u(t)\, dxdt + J_3^m + J_4^m + \frac{1}{2}\int_0^T \int_\Omega w(t)[v(t)]_x u(t)\, dxdt.
\end{aligned} \tag{5.61}$$

Since the inclusion $v \in L^\infty(0,T;V_\delta)$ implies $v \in L^\infty(Q)$, it follows that $v(\cdot)u(\cdot) \in L^2(0,T;H)$. Hence,

$$J_1^m \to 0 \ \text{ as } \ m \to \infty \ \text{ by condition (5.54).} \tag{5.62}$$

Besides, the estimate

$$\int_0^T \|[v(t)]_x\|_H \|u(t)\|_H \, dt \le \delta_0^{-1/2} \int_0^T \|[v(t)]_x\|_{V_\delta} \|u(t)\|_H \, dt$$
$$\le \delta_0^{-1/2} \|v\|_{L^2(0,T;V_\delta)} \|u\|_{L^2(0,T;H)} \le \sqrt{\delta_0^{-1} T}\, \|v\|_{L^\infty(0,T;V_\delta)} \|u\|_{L^2(0,T;V_0)}$$

shows that $[v(\cdot)]_x u(\cdot) \in L^1(0,T;H)$. Therefore, from (5.55), we deduce that

$$J_1^m \to 0 \ \text{ as } \ m \to \infty \ \text{ by condition (5.54).} \tag{5.63}$$

Further we notice that $\{[w_m(\cdot)]_x u(\cdot)\}_{m\in\mathbb{N}}$ is a bounded sequence in $L^1(0,T;H)$. Indeed,

$$\int_0^T \|[w_m(t)]_x u(t)\|_H \, dt \le \|[w_m]_x\|_{L^2(0,T;H)} \|u\|_{L^2(0,T;H)}$$
$$\le \|u\|_{L^2(0,T;V_0)} \left[\sup_{m\in\mathbb{N}} \|w_m\|_{L^2(0,T;V_0)} \right] \overset{\text{by (5.51)}}{\le} C < +\infty. \tag{5.64}$$

Taking into account the characterization of the set $W^{1,\infty}(0,T;V_\delta)$ (see Evans [7, p.279]), we have: $\{v_m(\cdot)\}_{m\in\mathbb{N}}$ is a bounded sequence in the space of Lipschitz continuous functions $C^{0,1}([0,T];V_\delta)$. Hence, by Arzelà–Ascoli theorem, we obtain

$$\lim_{m\to\infty} |J_2^m| \le \lim_{m\to\infty} \int_0^T \|[w_m(t)]_x u(t)\|_H \, \|v_m(t) - v(t)\|_H \, dxdt$$
$$\overset{\text{by (5.64)}}{\le} C \lim_{m\to\infty} \|v_m - v\|_{C([0,T];H)} \le C \lim_{m\to\infty} \|v_m - v\|_{C([0,T];V_\delta)} = 0. \tag{5.65}$$

It remains to study the asymptotic behaviour of the forth term J_4^m. To this end, we make use of the following chain of estimates

$$\begin{aligned} 2\,|J_4^m| &\le \int_0^T \|w_m(t)\|_{L^\infty(\Omega)} \, \|[v_m(t)]_x - [v(t)]_x\|_H \, \|u(t)\|_H \, dt \\ &\overset{\text{by (2.5)}}{\le} C_A \int_0^T \|w_m(t)\|_H^{1/2} \|w_m(t)\|_{V_0}^{1/2} \, \|[v_m(t)]_x - [v(t)]_x\|_H \, \|u(t)\|_H \, dt \\ &\le \underbrace{C_A \sup_{m\in\mathbb{N}} \|w_m\|^{1/2}_{L^\infty(0,T;H)}}_{D} \int_0^T \|w_m(t)\|_{V_0}^{1/2} \, \|[v_m(t)]_x - [v(t)]_x\|_H \, \|u(t)\|_H \, dt \\ &\le D \|u\|_{L^2(0,T;H)} \left(\int_0^T \|w_m(t)\|_{V_0} \, \|[v_m(t)]_x - [v(t)]_x\|_H^2 \, dt \right)^{1/2} \\ &\le D \|u\|_{L^2(0,T;V_0)} \left(\sup_{m\in\mathbb{N}} \|w_m\|_{L^2(0,T;V_0)} \right)^{1/2} \left(\int_0^T \|[v_m(t)]_x - [v(t)]_x\|_H^4 \, dt \right)^{1/4}. \end{aligned} \tag{5.66}$$

Since $\sup_{m\in\mathbb{N}} \|w_m\|_{L^2(0,T;V_0)} \overset{\text{by (5.54)}}{<} +\infty$ and

$$\int_0^T \|[v_m(t)]_x - [v(t)]_x\|_H^4 \, dt \le \delta_0^{-2} \int_0^T \left(\int_\Omega |[v_m(t)]_x - [v(t)]_x|^2 \, \delta \, dx \right)^2 dt$$
$$\le \delta_0^{-2} \int_0^T \|v_m(t) - v(t)\|_{V_\delta}^4 \, dt = \delta_0^{-2} \|v_m - v\|^4_{L^4(0,T;V_\delta)} \overset{\text{by (5.57)}}{\to} 0 \text{ as } m \to \infty,$$

it follows from (5.66) that

$$J_4^m \to 0 \quad \text{as} \quad m \to \infty. \tag{5.67}$$

Thus, in view of the properties (5.62), (5.63), (5.65), and (5.67), the limit passage in (5.61) as $m \to \infty$ leads to the desired relation (5.60). The proof is complete. □

PROPOSITION 5.7. *Let* $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$ *be a sequence with properties* (5.51)–(5.53) *and* (5.54)–(5.59), *and let* $u \in L^2(0,T;V_\delta)$ *be an arbitrary distribution. Then*

$$\lim_{m\to\infty} \int_0^T b_2(v_m(t), v_m(t), u(t))\, dt = \int_0^T b_2(v(t), v(t), u(t))\, dt. \tag{5.68}$$

Proof. Taking into account the definition of the trilinear form b_2 (see (3.9) and (3.11)), we have

$$\begin{aligned}
&\int_0^T b_2(v_m(t), v_m(t), u(t))\, dt = \int_0^T \int_\Omega v_m(t)[v_m(t)]_x u(t)\, dxdt \\
&\quad = \int_0^T \int_\Omega (v_m(t) - v(t))\, [v(t)]_x u(t)\, dxdt + \int_0^T \int_\Omega v_m(t)\left([v_m(t)]_x - [v(t)]_x\right) u(t)\, dxdt \\
&\quad + \int_0^T \int_\Omega v(t)[v(t)]_x u(t)\, dxdt = D_1^m + D_2^m + \int_0^T b_2(v(t), v(t), u(t))dt.
\end{aligned} \tag{5.69}$$

Then, in view of implication $v \in L^\infty(0,T;V_\delta) \;\Rightarrow\; v \in L^\infty(Q)$, we obtain

$$\begin{aligned}
\Big| \int_0^T \int_\Omega (v_m(t) - v(t))\, [v(t)]_x u(t)\, dxdt \Big| &\le \|u\|_{L^\infty(Q)} \int_0^T \|v_m(t) - v(t)\|_H \|u(t)\|_H\, dt \\
&\le \|v\|_{L^\infty(Q)} \|u\|_{L^2(0,T;H)} \|v_m - v\|_{L^2(0,T;H)} \\
&\le \delta_0^{-1} \|v\|_{L^\infty(Q)} \|u\|_{L^2(0,T;V_\delta)} \|v_m - v\|_{L^2(0,T;V_\delta)}.
\end{aligned}$$

Hence,

$$\lim_{m\to\infty} D_1^m = 0 \text{ by the strong convergence (5.57).} \tag{5.70}$$

As for the term D_2^m, we have

$$\begin{aligned}
|D_2^m| &= \Big| \int_0^T \int_\Omega v_m(t)\left([v_m(t)]_x - [v(t)]_x\right) u(t)\, dxdt \Big| \\
&\le \int_0^T \|v_m(t)\|_{L^\infty(\Omega)} \|[v_m(t)]_x - [v(t)]_x\|_H \|u(t)\|_H\, dt \\
&\overset{\text{by (2.1), (2.4)}}{\le} 2\sqrt{\delta_0^{-1}} \|v_m(t)\|_{L^\infty(0,T;V_\delta)} \int_0^T \|[v_m(t)]_x - [v(t)]_x\|_H \|u(t)\|_H\, dt \\
&\le \underbrace{2\sqrt{\delta_0^{-1}} \sup_{m\in\mathbb{N}} \|v_m(t)\|_{L^\infty(0,T;V_\delta)}}_{C} \|[v_m]_x - [v]_x\|_{L^2(0,T;H)} \|u\|_{L^2(0,T;H)} \\
&\le C\delta_0^{-1} \|v_m - v\|_{L^2(0,T;V_\delta)} \|u\|_{L^2(0,T;V_\delta)} \longrightarrow 0 \text{ as } m \to \infty \text{ by (5.57).}
\end{aligned} \tag{5.71}$$

Combining this result with (5.70), we can pass to the limit in (5.69). Thus, the equality (5.69) follows. □

We are now in a position to prove the main result of this section.

THEOREM 5.8. *Assume* (3.1)–(3.2) *hold true. Let* $\{(w_m(t)+\eta^*, u_m(t)+u^*(t))\}_{m\in\mathbb{N}}$ *be a sequence of Galerkin approximations to the corresponding weak solution of the initial-boundary value problem* (3.3)–(3.5). *Then there exists a unique pair*

$$(w,v)\in[L^\infty(0,T;H)\cap W_0(0,T)]\times W^{1,\infty}(0,T;V_\delta)\quad \textit{with}\quad \dot w\in L^\infty(0,T;V_0^*)$$

such that (w,v) *is a limit for the entire sequence* $\{(w_m(t),v_m(t))\}_{m\in\mathbb{N}}$ *as* $m\to\infty$ *in the following sense*

$$w_m \rightharpoonup w \text{ weakly in } L^2(0,T;V_0), \tag{5.72}$$

$$w_m \overset{*}{\rightharpoonup} w \text{ weakly-}* \text{ in } L^\infty(0,T;H), \tag{5.73}$$

$$\dot w_m \rightharpoonup \dot w \text{ weakly in } L^2(0,T;V_0^*) \text{ and weakly-}* \text{ in } L^\infty(0,T;V_0^*), \tag{5.74}$$

$$v_m \to v \text{ strongly in } L^p(0,T;V_\delta) \text{ for any } p\in(1,\infty) \text{ by } (5.53), \tag{5.75}$$

$$v_m \overset{*}{\rightharpoonup} v \text{ weakly-}* \text{ in } L^\infty(0,T;V_\delta), \tag{5.76}$$

$$\dot v_m \rightharpoonup \dot v \text{ weakly in } L^2(0,T;V_\delta) \text{ and weakly-}* \text{ in } L^\infty(0,T;V_\delta), \tag{5.77}$$

and $(w+\eta^*, v+u^*)$ *is the weak solution to the initial-boundary value problem* (3.3)–(3.5) *in the sense of Definition 5.1.*

Proof. To begin with, we note that the sequence of Galerkin approximations is relative compact with respect to the convergence (5.72)–(5.77) (see Remark 5.4 for the details). Let $\{(w_m(t),v_m(t))\}_{m\in\mathbb{N}}$ be its arbitrary subsequence with properties (5.72)–(5.77). Our first intension is to use these properties in order to pass to the limit as $m\to\infty$ in variational problem (5.5)–(5.6). However, we have to keep in mind that the test functions in (5.5)–(5.6) have to be chosen in $V_{0,m}\times V_{\delta,m}$. To do so, we fix a couple of distributions $(z,u)\in L^2(0,T;V_0)\times L^2(0,T;V_\delta)$. Since

$$z(t)=\sum_{k=1}^{\infty}\omega_k(t)\zeta_k \quad\text{and}\quad u(t)=\sum_{k=1}^{\infty}\lambda_k(t)\xi_k$$

and these series are convergent in V_0 and V_δ, respectively, for a.e. $t\in[0,T]$, we set

$$z_N(t)=\sum_{k=1}^{N}\omega_k(t)\zeta_k \quad\text{and}\quad u_N(t)=\sum_{k=1}^{N}\lambda_k(t)\xi_k$$

and keep N fixed, for the time being. If $m>N$, then we obviously have $(z_N,u_N)\in L^2(0,T;V_{0,m})\times L^2(0,T;V_{\delta,m})$. Hence, multiplying equation (5.5) by $\omega_k(t)$ and equation (5.6) by $\lambda_k(t)$ and summing for $k=1,\dots,m$, in view of the property (5.7), after integration over $(0,T)$, we get

$$\begin{aligned}\int_0^T (\dot w_m(t), z_N(t))_H\,dt &+ \int_0^T a_1(w_m(t),z_N(t))\,dt + \int_0^T b_1(w_m(t),v_m(t),z_N(t))\,dt\\ &+\int_0^T b_1(w_m(t),u^*(t),z_N(t))\,dt\\ &+\frac12\int_0^T \left(r_0\,[v_m(t)]_x+\eta^*\,[v_m(t)]_x, z_N(t)\right)_H\,dt\\ &+\frac12\int_0^T \left([r_0+\eta^*]\,[g(t)-h(t)], z_N(t)\right)_H\,dt=0,\end{aligned} \tag{5.78}$$

$$\begin{aligned}
\int_0^T (\dot v_m(t), u_N(t))_H\, dt &+ \int_0^T a_2(\dot v_m(t), u_N(t))\, dt + \int_0^T b_2(v_m(t), v_m(t), u_N(t))\, dt\\
&+ \int_0^T (\mu(t)\, [w_m(t)]_x\,, u_N(t))_H\, dt + \int_0^T ([v_m(t)]_x\, u_x^*(t), u_N(t))_H\, dt\\
&= \int_0^T (f(t), u_N(t))_H\, dt - \int_0^T (\dot u^*(t), u_N(t))_H\, dt\\
&- \int_0^T b_2(u^*(t), u^*(t), u_N(t))\, dt. \qquad (5.79)
\end{aligned}$$

Due to the weak convergence of the sequences $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$ and $\{(\dot w_m(t), \dot v_m(t))\}_{m\in\mathbb{N}}$ in their respective spaces (see (5.72)–(5.77) for the details), we can pass to the limit as $m\to\infty$ in that relations. Since (see (5.7),(5.74), and (5.77))

$$\begin{aligned}
&(\dot w_m(t), z_N(t))_H = \langle \dot w_m(t), z_N(t)\rangle_{V_0^*;V_0} \stackrel{m\to\infty}{\longrightarrow} \langle \dot w(t), z_N(t)\rangle_{V_0^*;V_0}\,,\\
&(\dot v_m(t), u_N(t))_H + a_2(\dot v_m(t), u_N(t))\\
&\quad = \langle \dot v_m(t), u_N(t)\rangle_{V_\delta^*;V_\delta} + a_2(\dot v_m(t), u_N(t)) \stackrel{m\to\infty}{\longrightarrow} \langle \dot v(t), u_N(t)\rangle_{V_\delta^*;V_\delta} + a_2(\dot v(t), u_N(t)),
\end{aligned}$$

it follows from Propositions 5.6 and 5.7 that the following equalities are valid

$$\begin{aligned}
\int_0^T (\dot w(t), z_N(t))_H\, dt &+ \int_0^T a_1(w(t), z_N(t))\, dt + \int_0^T b_1(w(t), v(t), z_N(t))\, dt\\
&+ \int_0^T b_1(w(t), u^*(t), z_N(t))\, dt + \frac{1}{2}\int_0^T ((r_0+\eta^*)\,[u(t)]_x\,, z_N(t))_H\, dt\\
&+ \frac{1}{2}\int_0^T ([r_0+\eta^*]\,[g(t)-h(t)]\,, z_N(t))_H\, dt = 0, \qquad (5.80)\\
\int_0^T (\dot v(t), u_N(t))_H\, dt &+ \int_0^T a_2(\dot v(t), u_N(t))\, dt + \int_0^T b_2(v(t), v(t), u_N(t))\, dt\\
&+ \int_0^T (\mu(t)\,[w(t)]_x\,, u_N(t))_H\, dt + \int_0^T ([v(t)]_x\, u_x^*(t), u_N(t))_H\, dt\\
&= \int_0^T (f(t), u_N(t))_H\, dt - \int_0^T (\dot u^*(t), u_N(t))_H\, dt\\
&- \int_0^T b_2(u^*(t), u^*(t), u_N(t))\, dt. \qquad (5.81)
\end{aligned}$$

Now, we can let $N\to\infty$ keeping in mind that $z_N\to z$ and $u_N\to u$ strongly in $L^2(0,T;V_0)$ and $L^2(0,T;V_\delta)$, respectively. Taking into account that the result is valid for all $(z,u)\in L^2(0,T;V_0)\times L^2(0,T;V_\delta)$, we finally conclude the fulfilment of the following equalities

$$\begin{aligned}
&\langle \dot w(t), z\rangle_{V_0^*;V_0} + a_1(w(t), z) + b_1(w(t), v(t), z) + b_1(w(t), u^*(t), z)\\
&\qquad + \frac{1}{2}\,((r_0+\eta^*)\,[v(t)]_x\,, z)_H + \frac{1}{2}\,([r_0+\eta^*]\,[g(t)-h(t)]\,, z)_H = 0, \qquad (5.82)\\
&\langle \dot v(t), u\rangle_{V_\delta^*;V_\delta} + a_2(\dot v_m(t), u) + b_2(v(t), v(t), u) + (\mu(t)\,[w(t)]_x\,, u)_H\\
&\qquad + ([v(t)]_x\, u_x^*(t), u)_H = (f(t), u)_H - (\dot u^*(t), u)_H - b_2(u^*(t), u^*(t), u) \qquad (5.83)
\end{aligned}$$

for all $z\in V_0$ and $u\in V_\delta$, and almost each $t\in[0,T]$.

It remains to show that $(w(t), v(t))$ satisfy the initial conditions $w(0) = \eta_0 - \eta^*$ in H and $v(0) = u_0$ in V_δ. To begin with, we note that $w \in C([0,T]; H)$ and $v \in C([0,T]; V_\delta)$ (see Remark 2.3 and Lemma 2.1). Let us check the condition $v(0) = u_0$ in V_δ (the similar assertion for $w(0)$ can be verified in the same way). With that in mind we fix $u \in C^1([0,T]; V_\delta)$ with $u(T) = 0$ and apply the integration by parts in the following relations

$$\int_0^T \left(\dot{v}_m(t), u(t)\right)_H \, dt = -\left(U_m, u_N(0)\right)_H - \int_0^T \left(v_m(t), \dot{u}(t)\right)_H \, dt,$$
$$\int_0^T a_2\left(\dot{v}_m(t), u(t)\right) \, dt = -\left(U_m, u_N(0)\right)_{L^2(\Omega, \delta\, dx)} - \int_0^T a_2\left(v_m(t), \dot{u}(t)\right) \, dt.$$

Then from (5.81) we find

$$\begin{aligned}
-\int_0^T \left(v_m(t), \dot{u}_N(t)\right)_H \, dt &- \int_0^T a_2(v_m(t), \dot{u}_N(t)) \, dt + \int_0^T b_2(v_m(t), v_m(t), u_N(t)) \, dt \\
&+ \int_0^T \left(\mu(t) \left[w_m(t)\right]_x, u_N(t)\right)_H \, dt + \int_0^T \left(\left[v_m(t)\right]_x u_x^*(t), u_N(t)\right)_H \, dt \\
&= \int_0^T \left(f(t), u_N(t)\right)_H \, dt - \int_0^T \left(\dot{u}^*(t), u_N(t)\right)_H \, dt \\
&- \int_0^T b_2(u^*(t), u^*(t), u_N(t)) \, dt + \left(U_m, v_N(0)\right)_H + \left(U_m, v_N(0)\right)_{V_\delta}.
\end{aligned} \tag{5.84}$$

Letting the first $m \to \infty$ and then $N \to \infty$ and taking into account that $U_m \to u_0$ strongly in V_δ, we get from (5.84)

$$\begin{aligned}
-\int_0^T \left(v(t), \dot{u}(t)\right)_H \, dt &- \int_0^T a_2(v(t), \dot{u}(t)) \, dt + \int_0^T b_2(v(t), v(t), u(t)) \, dt \\
&+ \int_0^T \left(\mu(t) \left[w(t)\right]_x, u(t)\right)_H \, dt + \int_0^T \left(\left[v(t)\right]_x u_x^*(t), u(t)\right)_H \, dt \\
&= \int_0^T \left(f(t), u(t)\right)_H \, dt - \int_0^T \left(\dot{u}^*(t), u(t)\right)_H \, dt \\
&- \int_0^T b_2(u^*(t), u^*(t), u(t)) \, dt + \left(u_0, u(0)\right)_H + \left(u_0, u(0)\right)_{V_\delta}.
\end{aligned} \tag{5.85}$$

On the other hand, if we apply the integration by part formula to the relation (3.18) with $\psi = v(t)$, we obtain

$$\begin{aligned}
-\int_0^T \left(v(t), \dot{u}(t)\right)_H \, dt &- \int_0^T a_2(v(t), \dot{u}(t)) \, dt + \int_0^T b_2(v(t), v(t), u(t)) \, dt \\
&+ \int_0^T \left(\mu(t) \left[w(t)\right]_x, u(t)\right)_H \, dt + \int_0^T \left(\left[v(t)\right]_x u_x^*(t), u(t)\right)_H \, dt \\
&= \int_0^T \left(f(t), u(t)\right)_H \, dt - \int_0^T \left(\dot{u}^*(t), u(t)\right)_H \, dt \\
&- \int_0^T b_2(u^*(t), u^*(t), u(t)) \, dt + \left(v(0), u(0)\right)_H + \left(v(0), u(0)\right)_{V_\delta}.
\end{aligned} \tag{5.86}$$

Substracting (5.86) from (5.85), we have

$$(v(0), u(0))_H + (v(0), u(0))_{V_\delta} = (u_0, u(0))_H + (u_0, u(0))_{V_\delta}.$$

By arbitrariness of $u(0)$, we finally obtain $v(0)$ is equal to u_0 as elements of Hilbert space V_δ. It is clear that the similar assertion is valid for the equality $w(0) = \eta_0 - \eta^*$.

Thus, summing up the obtained results we can give the following conclusion: the pair $(w(t)+\eta^*, v(t)+u^*(t))$ is a weak solution to the initial-boundary value problem (3.3)–(3.5) in the sense of Definition 5.1. Since this conclusion is valid for any cluster pair of the sequence of Galerkin approximations $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$ and the system (3.3)–(3.5) admits a unique weak solution (see Lemma 4.1), it follows that (w, v) is a limit pair for the entire sequence $\{(w_m(t), v_m(t))\}_{m\in\mathbb{N}}$. □

REMARK 5.5. *It remains to observe that estimates* (5.13)–(5.14)*,* (5.38) *and* (5.42)–(5.43) *are still valid for the weak solution to the initial-boundary value problem* (3.3)–(3.5) $(w + \eta^*, v + u^*)$*. With that in mind it is enough to take into account the strong convergence* (5.4)*, the properties* (5.72)–(5.77)*, the lower semi-continuity of the norms* $\|\cdot\|_{L^2(0,T;V_\delta)}$*,* $\|\cdot\|_{L^2(0,T;V_0)}$*, and* $\|\cdot\|_{L^2(0,T;V_0^*)}$ *with respect to the weak convergence in the corresponding spaces, and pass to the limit in* (5.13)–(5.14)*,* (5.38) *and* (5.42)–(5.43) *as* $m \to \infty$.

6. On Regularity of Weak Solutions to the Boussinesq System. In the context of optimization problems closely related with the Boussinesq system, the regularity of the solutions of the corresponding initial-boundary value problem (3.3)–(3.5) plays a crucial role. Typically, the regularity of the solution improves with regularity of the original data. In view of this, we begin with the following result.

PROPOSITION 6.1. *In addition to* (3.2)*, let us assume that* $\eta_0 \in V_0 := H_0^1(\Omega)$*. Then a unique weak solution* $(w + \eta^*, v + u^*)$ *of the initial-boundary value problem* (3.3)–(3.5) *is such that*

$$w \in L^\infty(0,T;H) \cap L^2(0,T;H^2(\Omega) \cap V_0),\ \dot{w} \in L^2(0,T;H),\ v \in W^{1,\infty}(0,T;V_\delta) \tag{6.1}$$

and there exists a constant $D_* > 0$ *depending on*

$$\Omega,\ T,\ \nu,\ \|h_1\|_{L^\infty(0,T)},\ \|g_1\|_{L^\infty(0,T)},\ \|f\|_{L^\infty(0,T;H)},\ \|\sigma_0\|_{L^\infty(0,T)},\ \|\sigma_1\|_{L^\infty(0,T)},$$
$$\|\mu\|_{L^\infty(0,T;H)},\ \|\eta_0 - \eta^*\|_H,\ \|\eta_0\|^2_{V_0},\ \|u_0\|^2_{V_\delta},\ \|r_0\|_H,\ \delta_0,\quad \text{and}\quad \eta^*$$

which satisfies the estimates

$$\|w\|^2_{L^2(0,T;H^2(\Omega))} + \|w\|^2_{L^\infty(0,T;H)} + \|\dot{w}\|^2_{L^2(0,T;H)} \le D_*, \tag{6.2}$$
$$\|v\|^2_{L^\infty(0,T;V_\delta)} + \|\dot{v}\|^2_{L^\infty(0,T;V_\delta)} \le D_*. \tag{6.3}$$

Proof. Multiplying the equation (5.5) by $\dot{c}_k(t)$ and summing for $k = 1, \dots, m$, we get

$$\begin{aligned}\|\dot{w}_m(t)\|^2_H &+ a_1(w_m(t), \dot{w}_m(t)) + b_1(w_m(t), v_m(t), \dot{w}_m(t)) + b_1(w_m(t), u^*(t), \dot{w}_m(t))\\ &+ \frac{1}{2}\left((r_0 + \eta^*)\left[v_m(t)\right]_x, \dot{w}_m(t)\right)_H + + \frac{1}{2}\left((r_0 + \eta^*)\, u^*_x(t), \dot{w}_m(t)\right)_H = 0,\end{aligned} \tag{6.4}$$

for a.e. $t \in [0, T]$. We note that

$$a_1(w_m(t), \dot{w}_m(t)) = \nu\, (w_m(t), \dot{w}_m(t))_{V_0} = \frac{\nu}{2}\frac{d}{dt}\|[w_m(t)]_x\|^2_H \quad \text{for a.e. } t \in (0, T).$$

It was indicated in Remark 5.4 that, for each $m \in \mathbb{N}$, we have $v_m \in W^{1,\infty}(0,T;V_\delta)$. However, because of the continuous embedding $V_\delta \hookrightarrow C(\overline{\Omega})$ and the fact that inclusion $v_m \in W^{1,\infty}(0,T;V_\delta)$ implies $v_m \in C^{0,1}([0,T];V_\delta)$, we deduce: $[v_m]_x \in L^\infty(\Omega)$ for each $m \in \mathbb{N}$. Hence, there exists a constant $C_E > 0$ such that

$$\max_{t\in[0,T]} \|[v_m(t)]_x\|_{L^\infty(\Omega)} \le C_E \max_{t\in[0,T]} \|[v_m(t)]_x\|_{V_\delta} \le C_E \|v_m\|_{W^{1,\infty}(0,T;V_\delta)}.$$

As a result, by Hölder's and Young's inequalities, we derive the following estimate:

$$\begin{aligned}
&\frac{1}{2}\left(r_0\,[v_m(t)]_x + \eta^*\,[v_m(t)]_x\,, \dot{w}_m(t)\right)_H \le \frac{1}{2}\|r_0+\eta^*\|_H \|[v_m(t)]_x\|_{L^\infty(\Omega)}\|\dot{w}_m(t)\|_H \\
&\le \frac{C_E}{2}\left[\|r_0\|_H + \eta^*\right]\|v_m\|_{W^{1,\infty}(0,T;V_\delta)}\|\dot{w}_m(t)\|_H \\
&\overset{\text{by (4.6)}}{\le} \frac{C_E}{2}\left[\|r_0\|_H+\eta^*\right]\left[\frac{\varepsilon}{2}\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)} + \frac{1}{2\varepsilon}\|\dot{w}_m(t)\|^2_H\right] \\
&\qquad \left\{\text{letting } \ \varepsilon = \frac{3}{2}C_E\left[\|r_0\|_H+\eta^*\right]\right\} \\
&= \frac{3C_E^2}{8}\left[\|r_0\|_H+\eta^*\right]^2\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)} + \frac{1}{6}\|\dot{w}_m(t)\|^2_H.
\end{aligned} \tag{6.5}$$

The similar one holds true for the last term in (6.4)

$$\begin{aligned}
\frac{1}{2}\left(r_0 u_x^* + \eta^* u_x^*, \dot{w}_m(t)\right)_H &\le \frac{1}{2}\|r_0+\eta^*\|_H\|u_x^*\|_{L^\infty(\Omega)}\|\dot{w}_m(t)\|_H \\
&\le \frac{1}{2}\left[\|r_0\|_H+\eta^*\right]\underbrace{\left(\|g\|_{W_0^{1,\infty}(0,T)} + \|h\|_{W_0^{1,\infty}(0,T)}\right)}_{\widehat{C}}\|\dot{w}_m(t)\|_H \\
&\le \frac{1}{2}\left[\|r_0\|_H+\eta^*\right]\left[\frac{\varepsilon}{2}\widehat{C}^2 + \frac{1}{2\varepsilon}\|\dot{w}_m(t)\|^2_H\right]_{\left\{\varepsilon=\frac{3}{2}\left[\|r_0\|_H+\eta^*\right]\right\}} \\
&= \frac{3\widehat{C}^2}{8}\left[\|r_0\|_H+\eta^*\right]^2 + \frac{1}{6}\|\dot{w}_m(t)\|^2_H.
\end{aligned} \tag{6.6}$$

Since

$$b_1(w_m(t), v_m(t), \dot{w}_m(t)) = \int_\Omega \left[[w_m(t)]_x\, v_m(t)\dot{w}_m(t) + \frac{1}{2}w_m(t)[v_m(t)]_x\dot{w}_m(t)\right]\,dx,$$

it follows that

$$\begin{aligned}
&\left|b_1(w_m(t), v_m(t), \dot{w}_m(t))\right| \le \|v_m(t)\|_{L^\infty(\Omega)}\|[w_m(t)]_x\|_H\|\dot{w}_m(t)\|_H \\
&\quad + \frac{1}{2}\|w_m(t)\|_H\|[v_m(t)]_x\|_{L^\infty(\Omega)}\|\dot{w}_m(t)\|_H \\
&\overset{\text{by (2.4), (2.3)}}{\le} 2\sqrt{\delta_0^{-1}}\|v_m(t)\|_{V_\delta}\|w_m(t)\|_{V_0}\|\dot{w}_m(t)\|_H \\
&\quad + \frac{1}{2}C_E\|w_m(t)\|_{V_0}\|[v_m(t)]_x\|_{V_\delta}\|\dot{w}_m(t)\|_H \le \underbrace{\left[2\sqrt{\delta_0^{-1}} + \frac{1}{2}C_E\right]}_{\widetilde{C}_1}\|v_m\|_{W^{1,\infty}(0,T;V_\delta)} \\
&\quad \times \left[\frac{\varepsilon}{2}\|w_m(t)\|^2_{V_0} + \frac{1}{2\varepsilon}\|\dot{w}_m(t)\|^2_H\right]_{\varepsilon = 3\widetilde{C}_1\|v_m\|_{W^{1,\infty}(0,T;V_\delta)}} \\
&\le \frac{3\widetilde{C}_1^2}{2}\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)}\|w_m(t)\|^2_{V_0} + \frac{1}{6}\|\dot{w}_m(t)\|^2_H.
\end{aligned} \tag{6.7}$$

From this inequality and (6.4)–(6.6), we infer

$$\nu\frac{d}{dt}\|[w_m(t)]_x\|_H^2+\|\dot w_m(t)\|_H^2\le \underbrace{\frac{3C_E^2}{4}\left[\|r_0\|_H+\eta^*\right]^2}_{\widetilde C_2}\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)}$$

$$+\underbrace{\frac{3\widehat C^2}{4}\left[\|r_0\|_H+\eta^*\right]^2}_{\widetilde C_3}+3\widetilde C_1^2\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)}\|w_m(t)\|^2_{V_0}\quad\text{for a.e. } t\in(0,T).$$

Therefore, an integration over $(0,t)$ yields

$$\begin{aligned}
&\nu\|w_m(t)\|^2_{V_0}+\int_0^t\|\dot w_m(s)\|_H^2\,ds=\nu\|[w_m(t)]_x\|^2_H+\int_0^t\|\dot w_m(s)\|^2_H\,ds\\
&\le T\widetilde C_2\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)}+3\widetilde C_1^2\|v_m\|^2_{W^{1,\infty}(0,T;V_\delta)}\|w_m\|^2_{L^2(0,T;V_0)}+\nu\|\eta_0\|^2_{V_0}+\widetilde C_3T\\
&\le T\widetilde C_2\left[\sup_{m\in\mathbb N}\|v_m\|_{W^{1,\infty}(0,T;V_\delta)}\right]^2+\nu\|\eta_0\|^2_{V_0}+\widetilde C_3T\\
&+3\widetilde C_1^2\left[\sup_{m\in\mathbb N}\|v_m\|_{W^{1,\infty}(0,T;V_\delta)}\right]^2\left[\sup_{m\in\mathbb N}\|w_m\|_{L^2(0,T;V_0)}\right]^2\overset{\text{by }(5.51)\text{–}(5.53)}{<}+\infty.
\end{aligned}\tag{6.8}$$

As follows from this estimate, the sequences $\{w_m(\cdot)\}_{m\in\mathbb N}$ and $\{\dot w_m(\cdot)\}_{m\in\mathbb N}$ are bounded in $L^\infty(0,T;V_0)$ and $L^2(0,T;H)$, respectively. Hence, up to a subsequence, we can suppose that there exist appropriate subsequences (still denoted by the suffix m) such that (see (5.54), (5.56), and Remark 5.4)

$$w_m\overset{*}{\rightharpoonup}w\ \text{ in }\ L^\infty(0,T;V_0)\quad\text{and}\quad \dot w_m\rightharpoonup\dot w\ \text{ in }\ L^2(0,T;H)\ \text{ as } m\to\infty.\tag{6.9}$$

As a result, we can pass to the limit in (6.8) as $m\to\infty$ along a chosen subsequence and deduce by the weak lower semi-continuity of the norms in $L^\infty(0,T;V_0)$ and $L^2(0,T;H)$ that the same estimate holds for the limit element w. Hence,

$$w\in L^\infty(0,T;V_0)\quad\text{and}\quad \dot w\in L^2(0,T;H).\tag{6.10}$$

Moreover, in the similar spirit to the estimation like (6.7), it can be shown that

$$\left(w_x(t)v(t)+\frac12 w(t)v_x(t)\right)\in H\quad\text{for a.e. } t\in[0,T].\tag{6.11}$$

Indeed,

$$\begin{aligned}
\left\|w_x(t)v(t)+\frac12 w(t)v_x(t)\right\|_H&\le\|w_x(t)v(t)\|_H+\frac12\|w(t)v_x(t)\|_H\\
&\le\|w_x(t)\|_H\|v(t)\|_{L^\infty(\Omega)}+\frac12\|w(t)\|_{L^\infty(\Omega)}\|v_x(t)\|_H\\
&\overset{\text{by }(2.4)}{\le}2\sqrt{\delta_0^{-1}}\|w(t)\|_{V_0}\|v(t)\|_{V_\delta}+\|w(t)\|_{V_0}\|v_x(t)\|_H\\
&\le\underbrace{3\sqrt{\delta_0^{-1}}}_{2C}\|w(t)\|_{V_0}\|v(t)\|_{V_\delta}\\
&\le C\left[\|w(t)\|^2_{V_0}+\|v(t)\|^2_{V_\delta}\right]\overset{\text{by }(5.51)\text{–}(5.53)}{<}+\infty\quad\text{a.e. t([0,T].}
\end{aligned}\tag{6.12}$$

Taking this fact into account and combining it with property (6.10), we can rewrite the equation in (3.17) in the form

$$\nu\Big(w_x(t),\varphi_x\Big)_H = -\Big(\dot{w}(t) + w_x(t)\,(v(t)+u^*(t)) + \frac{1}{2}w(t)\,(v_x(t)+u_x^*(t))\,,\varphi\Big)_H$$
$$-\frac{1}{2}\Big(\,[(r_0+\eta^*)\,(v_x(t)+u_x^*(t))]\,,\varphi\Big)_H \quad \text{a.e. } t\in[0,T] \text{ for all } \varphi\in V_0.$$

Then, the regularity theory for elliptic equations (see [7]) implies that $w(t)\in H^2(\Omega)$ for a.e. $t\in[0,T]$ and

$$\|w(t)\|^2_{H^2(\Omega)} \le C(\nu,\Omega)\Big[\|\eta_0\|^2_{V_0} + \|\dot{w}(t)\|^2_H + \|w_x(t)\,(v(t)+u^*(t))\,\|^2_H$$
$$+\,\|w(t)\,(v_x(t)+u_x^*(t))\,\|^2_H + \|(r_0+\eta^*)\,(v_x(t)+u_x^*(t))\,\|^2_H\Big] \quad \text{a.e. } t\in[0,T].$$

Integrating this relation over $[0,T]$ and using (6.5), (6.8), and (6.12), we see that

$$\|(r_0+\eta^*)\,(v_x(t)+u_x^*(t))\,\|^2_{L^2(0,T;H)} \overset{\text{by (6.5)}}{\le} 2C_E^2\,[\|r_0\|_H+\eta^*]^2\left(\|u\|^2_{W^{1,\infty}(0,T;V_\delta)} + \widehat{C}^2T\right),$$
$$\|w\,(v_x(t)+u_x^*(t))\,\|^2_{L^2(0,T;H)} \overset{\text{by (6.12)}}{\le} 2\delta_0^{-1}\int_0^T \|w(t)\|^2_{V_0}\left(\|v(t)\|^2_{V_\delta} + \widehat{C}^2\|\delta\|_{L^1(\Omega)}\right)dt$$
$$\le 2\delta_0^{-1}\left(\|v\|^2_{W^{1,\infty}(0,T;V_\delta)} + \widehat{C}^2\|\delta\|_{L^1(\Omega)}\right)\|w\|^2_{L^2(0,T;V_0)},$$
$$\|w_x\,(v(t)+u^*(t))\,\|^2_{L^2(0,T;H)} \overset{\text{by (6.12)}}{\le} 4\delta_0^{-1}\int_0^T \|w(t)\|^2_{V_0}\left(\|v(t)\|^2_{V_\delta} + \widehat{C}^2\|\delta\|_{L^1(\Omega)}\right)dt$$
$$\le 4\delta_0^{-1}\left(\|v\|^2_{W^{1,\infty}(0,T;V_\delta)} + \widehat{C}^2\|\delta\|_{L^1(\Omega)}\right)\|w\|^2_{L^2(0,T;V_0)},$$
$$\|\dot{w}\|^2_{L^2(0,T;H)} \overset{\text{by (6.8)}}{\le} T\widetilde{C}_2\|v\|^2_{W^{1,\infty}(0,T;V_\delta)} + \nu\|\eta_0\|^2_{V_0} + \widetilde{C}_3T$$
$$+\,3\widetilde{C}_1^2\|v\|^2_{W^{1,\infty}(0,T;V_\delta)}\|w\|^2_{L^2(0,T;V_0)},$$

where $\widehat{C}$ is given by (5.23). This leads us to the conclusion: there exists a constant $C_* = C_*(\nu,\Omega,T,\delta_0)>0$ independent of w and u such that the following estimate

$$\|w\|^2_{L^2(0,T;H^2(\Omega))} \le C_*\Big[\|\eta_0\|^2_{V_0} + (\eta^*)^2 + \Big(1+\|w\|^2_{L^2(0,T;V_0)}\Big)\|v\|^2_{W^{1,\infty}(0,T;V_\delta)}$$
$$+\,\|g\|^2_{W_0^{1,\infty}(0,T)} + \|h\|^2_{W_0^{1,\infty}(0,T)}\Big] \tag{6.13}$$

holds true and, therefore, $w\in L^2(0,T;H^2(\Omega))$ by (6.13) and Theorem 5.8. Thus, (6.2) is a direct consequence of (6.13) and estimates (5.13)–(5.14), (5.38) and (5.42)–(5.43) which are the same for the functions w and u as it is indicated in Remark 5.5.

To end the proof, it remains to notice that the conclusion given above is valid for any cluster pair of the sequence of Galerkin approximations $\{(w_m(t),u_m(t))\}_{m\in\mathbb{N}}$. Since the system (3.3)–(3.5) admits a unique weak solution (see Lemma 4.1), it follows that the convergence (6.9) takes a place for the entire sequences $\{w_m(\cdot)\}_{m\in\mathbb{N}}$ and $\{\dot{w}_m(\cdot)\}_{m\in\mathbb{N}}$. $\square$

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